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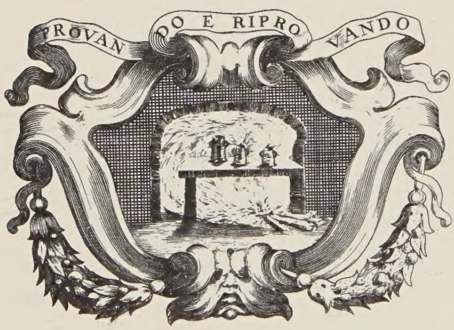

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Second-Order Effect of Spin-Orbit Interaction on the Paramagnetic Resonance Spectra of Ions with a Single $3d$ Electron.

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(ricevuto il 2 Gennaio 1961)

Summary. — In order to explain in a better way the covalent bonding factors for ions with a single $3d$ electron, the second-order g-factors, arising out of the mixing of orbital doublet states with orbital triplet states through spin-orbit interaction, have been revaluated for crystal fields of different symmetries. On the basis of these expressions, discussions have been made about the e.p.r. spectra of Ti^{3+} in some salts and of V^{4+} in TiO.

1. – Introduction.

Recently a detailed analysis of covalent bonding factors has been made for Ti^{3+} in some salts and for V^{4+} in TiO_2 [1,2]. It was found that in some cases these factors are unusually small. This was particularly true for V^{4+} in TiO_2 in which case $K_\perp$ was found to be less than 0.3 in order that $K_\parallel$ may be smaller than unity. Moreover in the case of Ti^{3+} in Al_2O_3, on the assumption of a positive Δ, the trigonal field splitting parameter, $g_\perp$ came out to be identically zero, whereas the experimental value was somewhere less than 0.1 and possibly not identically zero [3].

(*) The work was done while the author was in Moscow. Present address: Laboratorie de Physique Generale, Institut Fourier, Grenoble, France.

(1) D. K. RAY: *Žurn. Ėkps. Teor. Fiz.* (to be published).

(2) D. K. RAY: *Žurn. Ėkps. Teor. Fiz.* (to be published).

(3) L. S. KORNIENKO and A. M. PROKHOROV: *Žur. Ėxp. Teor. Fiz.*, **38**, 1651 (1960).

In all the above analyses, it was assumed that spin-orbit interaction is effective only among the lowest triplet states. It would be of interest to see if the second-order g-factors coming out of the effect of the higher orbital doublet states through spin-orbit interaction can explain the situation in a better way. This is particularly important due to the fact that the values of the covalent bonding factors depend sensitively on the anisotropy of the g-factors as well as their difference from the free-spin value.

2. – Expression for g-factors.

The energy levels and the eigenstates in the case of a cubic and a trigonal or tetragonal fields are given in Tables I and II, where W_0, W_1, etc., depend on the values of cubic and trigonal or tetragonal fields and the coefficients a, b, etc., depend on their relative values. If the trigonal field is very small, we can take $a \approx \sqrt{\frac{2}{3}}$ and $b \approx \sqrt{\frac{1}{3}}$. If we designate $W_0 - W_1 = \Delta$, then for Δ negative, the orbital singlet is lowest and reverse is the case for Δ positive.

TABLE I. – *States for trigonal field case.*

State	Energy
$\|0\rangle$	W_0
$\left.\begin{matrix} a\|-2\rangle + b\|1\rangle \\ a\|\ \ 2\rangle - b\|1\rangle \end{matrix}\right\}$	W_1
$\left.\begin{matrix} b\|-2\rangle - a\|1\rangle \\ b\|2\rangle + a\|-1\rangle \end{matrix}\right\}$	W_2

TABLE II. – *States for tetragonal field case.*

State	Energy
$\frac{1}{\sqrt{2}}(\|2\rangle - \|-2\rangle)$	W_0
$\left.\begin{matrix} \|\ \ 1\rangle \\ \|-1\rangle \end{matrix}\right\}$	W_1
$\frac{1}{\sqrt{2}}(\|2\rangle + \|-2\rangle)$	W_2
$\|0\rangle$	W_3

We shall first take up the case of trigonal field. In this case, we get the following expressions for first and second order g-factors:

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For Δ negative:

$$(1)\qquad \begin{cases} g_{\parallel}^{(1)} = 2\left[\sin^2\alpha - (1+P)\cos^2\alpha\right], \\ g_{\parallel}^{(2)} = 2\left[\dfrac{9a^2b^2\lambda}{W_2 - W_1} - \dfrac{3\sqrt{6}\,a^2b\lambda}{W_2 - W_0}\sin\alpha\cos\alpha\right], \end{cases}$$

and

$$(2)\qquad \begin{cases} g_{\perp}^{(1)} = 2\left[\sqrt{2}Q\sin\alpha\cos\alpha + \sin^2\alpha\right], \\ g_{\perp}^{(2)} = 2\left[3\sqrt{\dfrac{3}{2}}\,\dfrac{a^2b\lambda}{W_2 - W_1}\sin\alpha\cos\alpha - \dfrac{3a^2\lambda}{W_2 - W_0}\sin^2\alpha\right], \end{cases}$$

with

$$(3)\qquad \operatorname{tg}\alpha = -\frac{1}{\sqrt{2}\lambda_2}\left[\Delta + \frac{\lambda_1}{2} + \left(\Delta^2 + \frac{9}{4}\lambda_2^2 + \Delta\lambda_2\right)^{\frac{1}{2}}\right],$$

where,

$$(4)\qquad \begin{cases} p = \left(\dfrac{2}{3}a^2 + \dfrac{1}{3}b^2 + \dfrac{2\sqrt{2}}{3}ab\right)K_{\pi\pi} + \dfrac{4}{3}\left(a^2 - b^2 - \dfrac{ab}{\sqrt{2}}\right)K_{\pi\sigma}, \\ Q = -\left(\sqrt{\dfrac{2}{3}}\,a - \dfrac{2}{\sqrt{3}}\,b\right)K_{\pi\sigma} + \left(\sqrt{\dfrac{2}{3}}\,a + \dfrac{1}{\sqrt{3}}\,b\right)K_{\pi\pi}, \\ \lambda_1 = \left(\dfrac{2}{3}a^2 + \dfrac{1}{3}b^2 + \dfrac{2\sqrt{2}}{3}ab\right)\lambda R_{\pi\pi} + \dfrac{4}{3}\left(a^2 - b^2 - \dfrac{ab}{\sqrt{2}}\right)\lambda R_{\pi\sigma}, \\ \lambda_2 = -\left(\sqrt{\dfrac{2}{3}}\,a - \dfrac{2}{\sqrt{3}}\,b\right)\lambda R_{\pi\sigma} + \left(\sqrt{\dfrac{2}{3}}\,a + \sqrt{\dfrac{1}{3}}\,b\right)\lambda R_{\pi\pi}. \end{cases}$$

For Δ positive:

$$(5)\qquad \begin{cases} g_{\parallel}^{(1)} = 2(1-P) \\ g_{\parallel}^{(2)} = -\dfrac{18a^2b^2\lambda}{W_2 - W_1}, \end{cases}$$

and

$$(6)\qquad \begin{cases} g_{\perp}^{(2)} = 0, \\ g_{\perp}^{(1)} = -\dfrac{12ab\lambda}{W_2 - W_1}. \end{cases}$$

In the case of tetragonal field we get the following results:

For Δ negative:

$$(7)\qquad \begin{cases} g_{\parallel}^{(1)} = 2\sin^2\alpha - (1+K_{\parallel})\cos^2\alpha], \\ g_{\parallel}^{(2)} = 2\left[-\dfrac{4\lambda}{W_2 - W_0}\sin^2\alpha + \dfrac{2\sqrt{2}\lambda}{W_2 - W_1}\sin\alpha\cos\alpha\right], \end{cases}$$

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and

(8)
$$\begin{cases} g_{\perp}^{(1)} = 2[\sqrt{2}K_{\perp} \sin\alpha \cos\alpha + \sin^2\alpha] , \\ g_{\perp}^{(2)} = 2\left[\dfrac{\lambda}{W_2 - W_1}\cos^2\alpha - \dfrac{\sqrt{2}\lambda}{W_2 - W_0}\sin\alpha\cos\alpha\right] . \end{cases}$$

For Δ positive:

(9)
$$\begin{cases} g_{\parallel}^{(1)} = 2(1 - K_{\parallel}) , \\ g_{\parallel}^{(2)} = 0 , \end{cases}$$

and

(10)
$$\begin{cases} g_{\perp}^{(1)} = 0 , \\ g_{\perp}^{(2)} = -\dfrac{6\lambda}{W_2 - W_1} . \end{cases}$$

In the calculations of second-order terms, we have neglected the effect of covalent bonding as these terms are by themselves quite small and also we have,

(11)
$$\begin{cases} \langle\psi_m|\,\boldsymbol{L}\,|\psi_n\rangle = K_{mn}\langle\psi_m^d|\,\boldsymbol{L}\,|\psi_n^d\rangle \\ \langle\psi_m|\lambda\boldsymbol{L}\,|\psi_n\rangle = R_{mn}\langle\psi_m^d|\lambda\boldsymbol{L}\,|\psi_n^d\rangle . \end{cases}$$

3. – Analysis of paramagnetic resonance spectra of Ti^{3+} in some salts and of V^{4+} in TiO_2.

Analyses have already been made on the basis of first-order g-factors of Ti^{3+} in some salts and of V^{4+} in TiO_2 [1,2]. We shall see here how far some of these results change if we take account of second-order terms as well.

a) *Case of* Ti^{3+} *in alum* $(CsTi(SO_4)_2 \cdot 12\,H_2O)$. This is a case of trigonal field with Δ negative. From expressions (1) and (2), it is evident that there are three unknown constants P, Q and α corresponding to $g_{\parallel}$ and $g_{\perp}$, if we take the trigonal field to be small so that $W_2 - W_1 \approx W_2 - W_0 = 2 \cdot 10^4\ \text{cm}^{-1}$ [4], as given by optical absorption studies. So there can not be any unique values of these parameters. In Table III, some of the values of these quantities which satisfy the experimental g-values are given. By comparing these results with those given in (I), it is easy to find that the values of P and Q have become more anisotropic for a particular value of α. Moreover as Δ is small, P

[4] H. Hartmann and H. L. Schläfer: *Žur. Fiz. Kim.*, **197**, 116 (1951).

TABLE III. – *Case of* Ti^{3+} *in Alum* ($CsTi(SO_4)_2 \cdot 12H_2O$).

α	P	Q
110°	1.28	0.65
110° 30′	1.13	0.625
111°	0.99	0.60
111° 30′	0.86	0.58
112°	0.74	0.555
112° 30′	0.63	0.53
113°	0.52	0.512
113° 30′	0.42	0.49

and Q should be approximately equal to each other and this is so when $P = Q = 0.52$ in contrast to the value of 0.57 when only first order terms are considered.

b) *Case of* Ti^{3+} *in* Al_2O_3. It was shown in (1) that the experimental values of $g_{\parallel} = 1.067$ and $g_{\perp} < 0.1$ could be explained on the basis of Δ to be positive. It is evident from expression (6) that if we take only first order term, then $g_{\perp} = 0$. The extremely weak intensity of the line was explained on the basis of a small rhombic field. Now if we want to consider the effect of second-order terms, then we require the positions of orbital levels. In the absence of optical absorption data for Al_2O_3, there would not be much error if we took the same values for these as in the case of alum. So from the expression for $g_{\parallel}$, we get $p = 0.45$ and $g_{\perp}$ comes our as 0.04 on the assumption that $a \approx \sqrt{\frac{2}{3}}$ and $b \approx \sqrt{\frac{1}{3}}$. This value of $g_{\perp}$ is more in conformity with the experimental result. As the second-order terms come about only through orbital angular momentum, so the intensity of line spectra is not affected and the existence of a weak rhombic field has to be assumed for its explanation.

c) *Case of* V^{4+} *in* TiO_2. In this case the first-order terms gave extremely improbable values for the covalent bonding factors $K_{\parallel}$ and $K_{\perp}$. It was found that in order to fit the experimental data given by ZVEREV and PROKHOROV (5) or those of GERRITSEN and LEWIS (6), $K_{\parallel}$ becomes less than unity only for $K_{\perp} < 0.3$ and $K_{\parallel} = K_{\perp}$ when both are equal to0.26. Now to take account of second-order terms, the positions of energy levels are required, but these are not known for this case. As the radius of V^{4+} is somewhat different from that of Ti^{3+}, so it would not be correct to take these orbital splittings the same as in the case of Ti^{3+} in alum. But rough calculations show that both $K^{\parallel}$

(5) G. M. ZVEREV and A. M. PROKHOROV: *Žurn. Èkps. Teor. Fiz.*, **39**, 222 (1960).
(6) H. J. GERRITSEN and H. R. LAWIS: *Phys. Rev.*, **119**, 1010 (1960).

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and $K_{\perp}$ come out to be of the order of unity if we take probable values for the orbital splittings. Exact calculations can be made when optical absorption studies have been made for this salt.

4. – Conclusion.

It is evident from the above analysis that sometimes second-order g-factors have considerable effect on the magnitudes of covalent bonding factors and it is not fair to neglect these terms for a detailed analysis of paramegnatic resonance spectra of ions. In a future publication we intend to go into the case of V^{+3} in Al_2O_3.

* * *

In conclusion, the author expresses his gratitude to Prof. A. M. PROKHOROV for discussions and to L. S. KORNIENKO and G. M. ZVEREV for their interest in the work.

RIASSUNTO (*)

Per spiegare nel modo migliore i fattori covalenti di legame per ioni con un solo elettrone $3d$, si sono valutati per campi cristallini di diverse simmetrie i fattori g di secondo ordine, che sorgono dalla miscela di stati di doppietto orbitale con stati di tripletto orbitale tramite interazioni spin-orbita. Sulla base delle espressioni trovate, si discutono gli spettri e.p.r. del Ti^{3+} in alcuni sali e della V^{4+} nel TiO.

(*) *Traduzione a cura della Redazione.*

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Theory of Leptons - I (*).

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(ricevuto il 10 Febbraio 1961)

Summary. — A finite relativistic theory of four-fermion interactions is formulated; the theory involves as an essential ingredient the use of an indefinite metric. The problems of interpretation raised by the use of the indefinite metric are analyzed in relation to the observables and structure of many-particle states in quantum field theory; and the consistency of the interpretive postulate is demonstrated. The theory incidentally provides a *raison d'être* for the muon.

1. – Introduction.

Elementary particles exhibit an extended spectrum not only in their masses but also in their interaction properties. However, in the classification into the four groups of particles, namely photon, leptons, mesons and baryons the particles belonging to each group not only cover a well-defined range of the mass spectrum but also have fairly similar properties (1). As far as is known at present the photon has a « universal interaction » (with all charged particles); the leptons on the other hand take part both in universal electromagnetic interaction and also universal weak interaction (2). The mesons and baryons

(*) Supported in part by the U.S. Atomic Energy Commission.

(1) R. E. Marshak and E. C. G. Sudarshan: *Introduction to Elementary Particle Physics* (New York, 1961).

(2) By universal we mean an « all-or-none » characterization, *i.e.*, all charged particles interact with the same strength, but no neutral particle interacts directly. Similarly, in the weak interactions only charged currents interact, the neutral do not. For a more systematic discussion, see ref. (1).

take part in all interaction strong, weak and electromagnetic; but precisely because of the « strength » of their dominant interaction it is not possible to give a definitive answer to the question of a possible universal law of interaction for the strong interactions. In terms of our present understanding, the number of particles included in the elementary particle spectrum belonging to the baryon and meson classes is larger than is strictly necessary to account for the number of conserved quantities; and the question has often been raised whether some of these particles may be understood as composite objects so that the number of truly elementary particles is smaller. Stated more precisely, the question is whether one can introduce fewer elementary fields into the theory rather than introduce a distinct field associated with every particle: in principle, in a Lagrangian theory such questions can be meaningfully asked. But in practice there are two major difficulties: firstly in view of the « strength » of the interaction simple approximations are not adequate and one has to use rather ingenious computational tricks to get satisfactory results even in orthodox theory, and even more so in an attempt at answering fundamental questions. Secondly, the systematic development of the theory in a perturbation series gives divergent answers. In orthodox field theory there exists at least certain models in which these infinite quantities can be circumvented by a formal renormalization, but the mathematically ill-defined operations involved make it unclear whether the question as to which particles are elementary can still be meaningful in the renormalized theory.

However, as far the system of leptons and photon are concerned, though they have interactions with particles belonging to the baryon-meson system, thus forming only an « open system », for many purposes the lepton-photon system can be treated as if it is a « closed system ». The justification for such an approximate treatment derives mainly from the great success of quantum electrodynamics (not involving baryons and mesons) and of the weak universal $V-A$ interaction as applied to leptonic weak decays. In these cases the coupling strengths are sufficiently small for a perturbation expansion to be meaningful; so the first obstacle mentioned in the last paragraph does not arise for the photon-lepton system. But the standard divergences appear; and in a more troublesome manner as far as weak interactions are concerned since a four-fermion interaction is not « renormalizable ». Nevertheless, the calculations of the lowest order results involving weak interactions are in excellent agreement with experimental results just as in the case of electrodynamics. It then appears that there should be a formulation of the theory of weak and electromagnetic interactions of leptons somewhat different in principle in that the infinities no longer appear, but to lowest order the predictions are practically the same as given in the lowest order predictions of the usual theory.

The point of view that the formal introduction of an indefinite metric together with a corresponding interpretative postulate (defining the « physical

states ») supplies the additional physical principle needed to construct a finite theory of elementary particle interactions has been emphasized in an earlier paper ([3]). It is the purpose of this paper to implement the program for the weak interaction of leptons since no consistent theory of weak interactions exists at present.

In the following section the orthodox Lagrangian theory of leptonic weak four-fermion interaction is summarized with a view to establishing the notation and for easy comparison with the present theory which is introduced in Section **3**.

2. – Conventional theory of four-fermion interactions.

The conventional field theory for interacting leptons (with electromagnetic interactions omitted) is formulated in terms of the Lagrangian density ([4]):

$$\mathscr{L}(x) = \mathscr{L}_0(x) + G J_\lambda^+(x) J^\lambda(x)\,, \tag{1}$$

where $\rho = \gamma^\lambda p_\lambda$ and

$$\mathscr{L}_0(x) = \overline{\chi}\rho\chi + \overline{\psi}_1(\rho - m_1)\psi_1 + \overline{\psi}_2(\rho - m_2)\psi_2\,, \tag{2}$$

$$J_\lambda(x) = \overline{\psi}_1\gamma_\lambda(1+\gamma_5)\chi + \overline{\psi}_2\gamma_\lambda(1+\gamma_5)\chi \tag{3}$$

and the fields χ, ψ_1, ψ_2 refer to the neutrino, electron and muon fields and m_1, m_2 the observed electron and muon masses. We have included a four-component neutrino field (rather than a two-component field) since this gives a more uniform treatment of all the lepton fields, though in the conventional theory the negative chiral component has no interactions at all. These fields obey the anticommutation relations for equal times:

$$\left|\begin{array}{l} \{\chi^+(\boldsymbol{x}, t),\ \chi\,(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x}-\boldsymbol{x}')\,, \\ \{\psi_1^+(\boldsymbol{x}, t),\ \psi_1(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x}-\boldsymbol{x}')\,, \\ \{\psi_2^+(\boldsymbol{x}, t),\ \psi_2(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x}-\boldsymbol{x}')\,, \\ \text{all other anticommutators} = 0\,. \end{array}\right. \tag{4}$$

([3]) E. C. G. Sudarshan: *Quantum-mechanical systems with indefinite metric* - I, submitted to *Phys. Rev.*

([4]) E. C. G. Sudarshan and R. E. Marshak: *Proc. Padua-Venice Conference on Mesons and Newly Discovered Particles* (1957); *Phys. Rev.*, **109**, 1860 (1958); R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958); J. J. Sakurai: *Nuovo Cimento*: **7**, 649 (1958).

For this Lagrangian there are three constants of motion:

$$(5)\qquad \begin{cases} Q = \displaystyle\int \mathrm{d}^3x \, \{\psi_1^+(\boldsymbol{x}, t)\,\psi_1(\boldsymbol{x}, t) + \psi_2^+(\boldsymbol{x}, t)\,\psi_2(\boldsymbol{x}, t)\}\,, \\ L = \displaystyle\int \mathrm{d}^3x \, \{\chi^+(\boldsymbol{x}, t)\,\chi(\boldsymbol{x}, t) + \psi_1^+(\boldsymbol{x}, t)\,\psi_1(\boldsymbol{x}, t) + \psi_2^+(\boldsymbol{x}, t)\,\psi_2(\boldsymbol{x}, t)\}\,, \\ K = \displaystyle\int \mathrm{d}^3x \, \{\chi^+(\boldsymbol{x}, t)\,\gamma_5\,\chi(\boldsymbol{x}, t)\}\,. \end{cases}$$

All of these have integral eigenvalues and refer to the total charge, the lepton number and the neutrino chiral number respectively. The weak coupling constant G has the dimensions of an area. We have, in accordance with present experimental results, included only the universal chirality-invariant $V-A$ interaction through charged currents.

Unlike the other three-field (Yukawa) interactions, the four-fermion interaction can lead to physical scattering and decay processes in the first order (first Born approximation):

$$(6)\qquad \begin{cases} \mathrm{e} + \nu \to \mathrm{e} + \nu & \mathrm{e} + \bar{\nu} \to \mathrm{e} + \bar{\nu} \\ \mu + \nu \to \mu + \nu & \mu + \bar{\nu} \to \mu + \bar{\nu} \\ \mu + \nu \rightleftarrows \mathrm{e} + \nu & \mu + \bar{\nu} \rightleftarrows \mathrm{e} + \bar{\nu} \\ \mu + \bar{\mathrm{e}} \rightleftarrows \nu + \bar{\nu} \rightleftarrows \mathrm{e} + \bar{\mathrm{e}} & \mathrm{e} + \bar{\mu} \rightleftarrows \nu + \bar{\nu} \rightleftarrows \mu + \bar{\mu}\,, \end{cases}$$

$$(7)\qquad \mu \to \mathrm{e} + \nu + \bar{\nu}\,.$$

Of these interactions only the weak decay process (7) is experimentally tested; and here the detailed predictions regarding the energy spectrum, polarization correlations, etc., are in excellent agreement with the results of the lowest order calculations, except for small corrections. Most of these corrections have been quantitatively analyzed in terms of electromagnetic effects (5); the remainder, if any, has been the subject of several speculations concerning intermediate vector mesons (6). Since the calculation of the decay spectrum, etc., from the covariant transition matrix element for the decay reaction is available elsewhere in the literature we shall not present them here. The scattering processes (6) have not been experimentally investigated so far; in most cases

(5) Quantitative calculations have been made by S. Berman and T. Kinoshita. For a comprehensive review, see R. P. Feynman: *Proc. Tenth Annual Rochester Conference on High Energy Physics*, University of Rochester, (Rochester, 1960).

(6) See, for example, T. D. Lee: *Proc. Tenth Annual Rochester Conference on High Energy Physics*, University of Rochester, (Rochester, 1960).

they are masked by other dominant reactions (7). By the same token « all available experimental results » are consistent with the interaction introduced above.

It is interesting to note that with the specific choice of the $V-A$ interaction there are no self-energy effects in lowest order; the only possible fermion loop contribution shown in Fig. 1 vanishes by reasons of invariance.

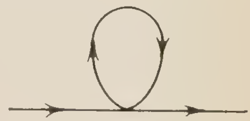


Fig. 1. – The first-order fermion self-energy diagram.

The agreement of the lowest order predictions with experimental results is gratifying especially in view of the smallness (8) of the coupling constant. However in any consistent theory one must be able to show that the higher order corrections are small; and one must be able, with sufficient labour, to compute these corrections. This is all the more important since there are small but significant deviations from the lowest order calculations (with electromagnetic corrections included) in the experimental results. These results may be summarized by saying that the effective four-fermion interaction is « slightly » non-local. Now a non-local effective interaction always results from any local interaction taken in higher orders. Some authors have considered this non-locality of the effective interaction as sufficient ground to interpret the four-fermion interaction as being mediated by the exchange of a vector meson, the basic interaction then being taken as the direct coupling of the vector meson field with the weak interaction current (3). We shall not do so, but rather consider the four-fermion interaction (1) itself to be the basic interaction; the problem then arises as to the magnitude of the higher order corrections stemming from this interaction.

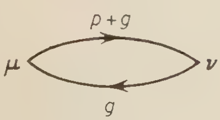


Fig. 2. – The two-vertex fermion loop.

It is however immediately verified that the expressions for higher order corrections are in general divergent and therefore meaningless as they stand. This divergence has its origin in the divergence of the two-vertex fermion loop illustrated in Fig. 2 with the corresponding matrix element proportional to the expression

$$\int d^4q \,\mathrm{Tr}\,\{\gamma^\mu(1+\gamma_5)(p+q-m)^{-1}\gamma^\nu(1+\gamma_5)(q-m')^{-1}\}\,,$$

(7) Recently H. M. CHIU has pointed out the possible relevance of the reaction $e+\bar{e}\to\nu+\bar{\nu}$ in stellar evolution; but no *experimental* information about the details of these processes (or even about their existence) is available to date.

(8) When we say that a coupling constant with the dimensions of an area is « small » we must include some natural unit of length; for this we take the inverse of the total energy in the center of mass for the process considered.

which is quadratically divergent. The lowest non-vanishing correction to the fermion self-energy as well as the lowest correction to the two-fermion scattering (or muon decay) processes involve this divergent quantity and hence cannot be computed without an essential change in the interpretative postulates of the theory. Such divergences are typical of all local interactions and have been handled in a covariant fashion through the renormalization technique in electrodynamics (and other renormalizable theories) by a redefinition of the physical parameters of renormalized coupling constant and masses. However such an elimination of the divergences is not possible for this theory in higher orders since new infinities appear in the higher orders and there are an infinite number of such terms. We shall see later that in the theory discussed in this paper also it is necessary to carry through a renormalization of the physical quantities but not for the purpose of elimination of the divergences. We must find then some other method of circumventing the divergences. A method of constructing such a finite theory was outlined in a previous paper [3]; and we shall discuss the theory of four-fermion interactions based on this method.

3. – Theory of coupled lepton fields.

The Lagrangian density (1) involves three different fermion fields χ, ψ_1, ψ_2. We now generalize this to contain eight fields, four charged fields and four neutral fields represented by $\psi^{(j)}$, $\chi^{(j)}$ respectively and write the analogues of (2) and (3) in the form

$$\mathscr{L}_0(x) = \sum_{j=1}^{4} \overline{\chi}^{(j)}(p - \mu^{(j)})\chi^{(j)} + \sum_{j=1}^{4} \overline{\psi}^{(j)}(p - m^{(j)})\psi^{(j)} , \tag{8}$$

$$J_\lambda(x) = \sum_{j=1}^{4}\sum_{j'=1}^{4} \overline{\psi}^{(j)}\gamma_\lambda(1+\gamma_5)\chi^{(j')} . \tag{9}$$

The eight parameters $\mu^{(j)}$ and $m^{(j)}$ have the dimensions of a mass but are not to be identified with any observed masses. These fields obey the following anticommutation relations for equal times:

$$\left\{\begin{aligned} &\{\chi^{+(j)}(\boldsymbol{x}, t),\ \chi^{(j')}(\boldsymbol{x}', t)\} = (-1)^{j+1}\delta_{jj'}\,\delta(\boldsymbol{x}-\boldsymbol{x}') ,\\ &\{\psi^{+(j)}(\boldsymbol{x}, t),\ \psi^{(j')}(\boldsymbol{x}', t)\} = (-1)^{j+1}\delta_{jj'}\,\delta(\boldsymbol{x}-\boldsymbol{x}') ,\\ &\text{all other anticommutators} = 0 . \end{aligned}\right. \tag{10}$$

One notes that there are two constants of motion for this interaction

$$\left\{\begin{aligned} Q &= \int \mathrm{d}^3x \sum_{j=1}^{4} \psi^{(j)+}(\boldsymbol{x}, t)\,\psi^{(j)}(\boldsymbol{x}, t) ,\\ L &= \int \mathrm{d}^3x \sum_{j=1}^{4} [\psi^{(j)+}(\boldsymbol{x}, t)\,\psi^{(j)}(\boldsymbol{x}, t) + \chi^{(j)+}(\boldsymbol{x}, t)\,\chi^{(j)}(\boldsymbol{x}, t)] . \end{aligned}\right. \tag{11}$$

The indefinite sign for the anticommutators in (10) shows that the metric in the space cannot be positive definite: more specifically, the fields $\chi^{(2)}$, $\chi^{(4)}$, $\psi^{(2)}$, $\psi^{(4)}$ all have the « wrong » sign for their anticommutators. The metric appropriate to this quantization (9) is given by

$$(12) \qquad \eta = \exp\left[\pi i\int \mathrm{d}^3x\,[\psi^{(2)+}\psi^{(2)} + \psi^{(4)+}\psi^{(4)} + \chi^{(2)+}\chi^{(2)} + \chi^{(4)+}\chi^{(4)}\right].$$

Here the definitions of χ^+ and $\overline{\chi}$, etc., are made in accordance with this metric; more specifically χ^+ is the pseudo-hermitian adjoint of χ. With these definitions if G is chosen real it then follows that $\mathscr{L}$ is pseudohermitian; and all expectation values of the pseudohermitian quantities being real, the Hamiltonian density constructed from the new Lagrangian density (as well as all hermitian functionals of it) would have real expectation values.

We now proceed in the usual fashion (10) to construct a perturbation series which is manifestly covariant, which is appropriate to the above interaction. Most of the steps in the derivation of the perturbation series are identical to the corresponding derivation for a theory with positive definite metric. We omit this derivation but merely state the intermediate result giving the S-matrix as a power series in the time-ordered products:

$$(13) \qquad S = \sum_{n=0}^{\infty}(-i)^n(n!)^{-1}\int \mathrm{d}^4x_1 \ldots \int \mathrm{d}^4x_n\, T\,\{\mathscr{H}(x_1)\ldots\mathscr{H}(x_n)\}\,,$$

where the co-ordinate space integrations run over the entire range $-\infty$ to $+\infty$, T is the time-ordering symbol, and $\mathscr{H}(x)$ is the interaction density in the interaction representation. For expressing S in terms of particle amplitudes, it is necessary to express the time-ordered product in terms of normal products and contraction functions (« propagators »). At this point the indefinite metric shows itself in giving a different expression for some of the contraction functions; one finds in fact that the « ghost fields » $\chi^{(2)}$, $\chi^{(4)}$, $\psi^{(2)}$, $\psi^{(4)}$ give rise to propogators with the opposite sign as compared to a normal field with the same mass.

The origin of this negative sign may be seen most clearly by working with a single Fermi oscillator with the « wrong » sign for the anticommutator; the creation and destruction operators a^+, a satisfy:

$$\{a, a^+\} = -1\,; \qquad \{a, a\} = \{a^+, a^+\} = 0\,.$$

(9) Compare S. N. Gupta: *Proc. Phys. Soc.*, **63**, 681 (1950); **64**, 850 (1951); K. Bleuler: *Helv. Phys. Acta*, **23**, 567 (1950).

(10) F. J. Dyson: *Phys. Rev.*, **75**, 1736 (1949).

The states $|0\rangle$ and $|1\rangle$ have the scalar products:

$$\langle 0|0\rangle = +1\,; \quad \langle 1|1\rangle = -1\,; \quad \langle 0|1\rangle = 0 = \langle 1|0\rangle\,.$$

Consequently the contraction function is given by

$$\langle 0\,|\,aa^+\,|\,0\rangle = \langle 0\,|\,\{a,\,a^+\} - a^+a\,|\,0\rangle = -1\,,$$

which is of the opposite sign to the usual case. Note that this result (like the corresponding result for the « normal » case) is independent of the representation. However, in writing down the complete matrix element it is also necessary to specify the representation chosen for the creation and destruction operators; for the « normal » case the usual choice is to take a, a^+ to be real and we shall choose the same convention for the present theory also. The « abnormal » fields would have a relative sign change between the emission and absorption matrix elements.

The rules for calculating transition amplitudes may then be written down using Feynman diagrams and the Feynman rules with two modifications:

i) all contraction functions involving the « abnormal » fields are to be taken with an additional negative sign;

ii) all emission matrix elements involving « abnormal » fields acquire an additional negative sign: all absorption matrix elements are unchanged.

As in conventional theory it is advantageous to work in momentum space; in the usual theory the fermion contraction function [11] is of the form $(\not{p} - m + i\varepsilon)^{-1}$. We mentioned in the last section that the abnormal fields acquired an additional negative sign to their contraction functions. But from (9) it follows that to every Feynman diagram there corresponds another Feynman diagram in which any neutral lepton line corresponding to any type is replaced by a neutral lepton line corresponding to any of the other three types; and similarly for the charged lepton lines. It then follows that, as far as the *internal* fermion lines are concerned only an effective neutral lepton contraction function

$$S_\chi(p) = \sum_{j=1}^{4} (-1)^{j+1} (\not{p} - \mu^{(j)} + i\varepsilon)^{-1} \tag{14a}$$

and an effective charged lepton contraction function

$$S_\psi(p) = \sum_{j=1}^{4} (-1)^{j+1} (\not{p} - m^{(j)} + i\varepsilon)^{-1} \tag{14b}$$

[11] This contraction function differs from the usual expression by a factor $i/(2\pi)^4$; we shall use this definition throughout this paper.

enter the theory. We may now choose the mass parameters such that

$$\mu^{(1)} - \mu^{(2)} + \mu^{(3)} - \mu^{(4)} = m^{(1)} - m^{(2)} + m^{(3)} - m^{(4)} = 0 \tag{15}$$

since they are so far undetermined. With this condition satisfied, the effective contraction functions decrease as fast as p^{-3} for large values of the momentum [12]. This decrease is sufficient to make the two-vertex fermion loop contribution (Fig. 2) finite; we shall discuss their precise evaluation later. We note in passing that this replacement of individual lines by an effective contribution applies to internal lines only. (The external lines have to be considered in a somewhat different fashion and involves a new interpretive postulate. These matters are discussed in Section **4**.) We shall assume in the rest of this paper that the requirement (15) is fulfilled by the masses.

Fig. 3. – The second-order fermion self-energy diagram.

Let us now consider some higher order effects. The simplest is the fermion self-energy and there is only one type of diagram which contributes to this which is shown in Fig. 3. By a procedure completely analogous to the one adopted in the usual theory [13] we can write for the self-energy of the charged and neutral leptons the matrices:

$$\left\{\begin{aligned} \Sigma_\psi(p) &= (2\pi)^{-8} G^2 \int \mathrm{d}^4 q\, \gamma_\mu(1+\gamma_5) S_\psi(p-q)\, \gamma_\nu(1+\gamma_5)\, C_\chi^{\mu\nu}(q)\,, \\ \Sigma_\chi(p) &= (2\pi)^{-8} G^2 \int \mathrm{d}^4 q\, \gamma_\mu(1+\gamma_5) S_\chi(p-q)\, \gamma_\nu(1+\gamma_5)\, C_\psi^{\mu\nu}(q)\,, \end{aligned}\right. \tag{16}$$

with

$$C_\chi^{\mu\nu}(q) = \int \mathrm{d}^4 k\, \mathrm{Tr}\,\{\gamma^\mu(1+\gamma_5) S_\chi(q+k)\, \gamma^\nu(1+\gamma_5) S_\chi(k)\}\,, \tag{17}$$

and similarly for $C_\psi^{\mu\nu}(q)$. Making use of the identity

$$(z-m_1)^{-1} + (z-m_3)^{-1} - \left(z + k - \frac{m_1+m_2}{2}\right)^{-1} - \left(z - k - \frac{m_1+m_2}{2}\right)^{-1} = \tag{18}$$

$$= 2 \int\limits_{(3m_1+m_2+k)/4}^{(m_1+3m_2-k)/4} \mathrm{d}\alpha \int\limits_{(m_1-m_2-k)/2}^{(m_2-m_1+k)/2} \mathrm{d}\beta\, (z-\alpha-\beta)^{-3}\,,$$

([12]) With only four fields with contraction functions of unit weight it is not possible to make the effective contraction functions fall off faster.

([13]) See, for example, J. M. Jauch and F. Rohrlich: *Theory of Photons and Electrons* (Cambridge, Mass., 1955).

we can rewrite the expression for $C^{\mu\nu}(q)$ in the form

$$C_{\chi}^{\mu\nu} = 16 \iint \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2 \iint \mathrm{d}\beta_1\, \mathrm{d}\beta_2\, B^{\mu\nu}_{\alpha_1+\beta_1,\, \alpha_2+\beta_2}(q)\,, \tag{19}$$

where the limits of the α and β integrations are as given above; and

$$B^{\mu\nu}_{\lambda_1,\,\lambda_2}(q) = \int \mathrm{d}^4k\, \frac{1}{4}\, \mathrm{Tr}\, \{\gamma^\mu(1+k_5)(q+k-\lambda_1+i\varepsilon)^{-3}\gamma^\nu(1+\gamma_5)(k-\lambda_2+i\varepsilon)^{-3}\} = \tag{20}$$

$$= \pi^2 i \int_0^1 \mathrm{d}x\, \{[\lambda_1^2 x + \lambda_2^2(1-x) - x\,(1-x)q^2]g^{\mu\nu} + 3x\,(1-x)\,[g^{\mu\nu} - q^\mu q^\nu]\}\cdot$$

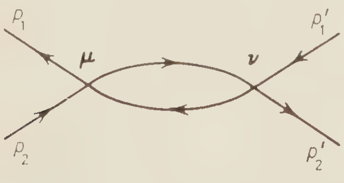
$$\cdot x^2(1-x)^2\,[\lambda_1^2 x + \lambda_2^2(1-x) - x(1-x)\,q^2]^{-4}\,.$$

The important point to note is that these quantities are finite; and for « reasonable » values of the masses, in view of the smallness of the coupling constant G, these expressions are « small ».

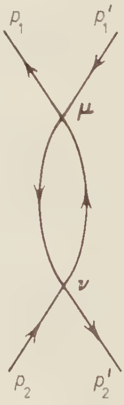
Similarly for the basic scattering process involving four external fermion lines the lowest order correction arises through the bubble diagram illustrated in Fig. 4. There are two such diagrams; the corresponding « reduced matrix element » (which replaces $Gg^{\mu\nu}$) is given by

$$A^{\mu\nu}(p_1, p_2, p_1', p_2') = (2\pi)^{-4} i G^2\, C^{\mu\nu}(p_1+p_2) \tag{21}$$

for the diagram Fig. 4*a* (and a similar expression for Fig. 4*b*). Here $C^{\mu\nu}$ is $C_{\chi}^{\mu\nu}$ for the choice of particles with momenta p_1, p_2 to be two charged leptons of opposite charge; and similar expressions for the other cases [14].



a)



b)

Fig. 4*a*, *b*. – Lowest-order corrections to scattering.

[14] In case p_1, p_2 are respectively charged and neutral, then the defining expression of $C^{\mu\nu}$ contains one S_ψ and one S_χ in an obvious manner.

One notes that these corrections are also finite. It is also of interest to note that this correction, eq. (21) leads to non-locality in the effective interaction; and the two-vertex loop, Fig. 2, here plays the role of an « intermediate vector meson ». There are two important differences however: the intermediate mesons are both charged and neutral, so that the theory does not correspond to a theory involving only charged vector mesons. The quantity $C^{\mu\nu}$ which plays the role of a vector meson contraction function is a non-trivial tensor in its indices; and corresponds to vector mesons with a continuum of masses. A consequence of this formal analogy is that the predictions of the present theory will be qualitatively similar to those of an intermediate vector meson theory.

One might calculate higher order effects in a similar fashion. Examples of some higher corrections in the next order are illustrated in Fig. 5. In a systematic analysis of these higher order corrections it is necessary to note

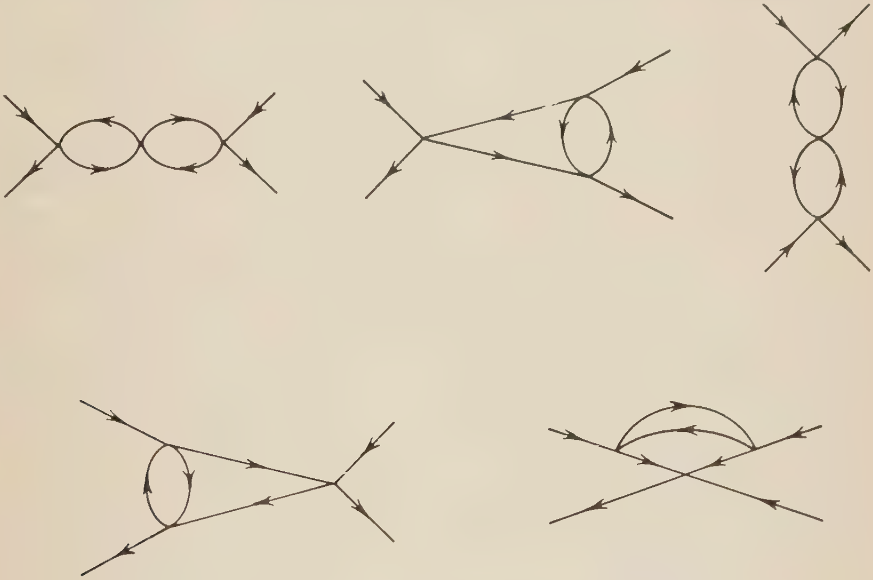


Fig. 5. – Some higher-order corrections to scattering.

that the « physical » lepton masses are shifted by varying amounts according to (16) and (17). In addition the physical particles are associated with « modified » fields; this feature is a consequence of the existence of several fields with all their « kinematical quantum numbers » the same and differing only in their mass parameters. To exhibit the theory in a form which is completely analogous to the conventional renormalized perturbation expansions [10] it is necessary to rewrite the perturbation series in terms of the « observed » masses, the « modified » fields and « observed » coupling constants. This renormali-

zation programme has of course nothing to do with the elimination of any infinite quantities; the renormalizations (or at least the coefficients of the power series expansion) are *finite* in the present theory. The manipulations are complicated and are relegated to the Appendix.

Two points are to be mentioned here. Firstly the weakness of the coupling and the consequent smallness of the expansion parameter makes most of these higher order corrections somewhat academic; but it is interesting to demonstrate how the theory could be consistently renormalized. Secondly, (as seen from (A.12)), the renormalized coupling constants are no longer equal; such a departure from universality of effective coupling strengths is of course expected, but again, in view of the smallness of the interaction constants, these departures will be small.

These remarks, then, dispose of the problem of higher order corrections to transition amplitudes; the transition amplitudes are now completely described once the external lines are specified. In view of the underlying indefinite metric, the transition amplitudes would correspond to a pseudounitary S-matrix. Hence the physical interpretation of such a theory requires a new interpretive postulate consistent with the present formalism as well as with general quantum mechanical principles. The nature of this new postulate was briefly discussed in two earlier papers. A more systematic discussion is given in the following sections.

4. – Measurable quantities in many-particle states.

In usual forms of quantum mechanics the states consist of vectors of unit length in a Hilbert space; if the dynamics is invariant under a group of transformations the states furnish a representation of the group which is in general reducible. These statements are true in all forms of quantum mechanics; in particular in all relativistic quantum theories the set of all states furnish a reducible representation of the proper inhomogeneous Lorentz group ([15]). If the relativistic quantum theory is a consistent theory and has an underlying Hilbert space (with a positive definite metric) these representations are unitary. For the inhomogeneous Lorentz group one can prove ([16]) that all reducible representations are the direct sum (or direct integral) of irreducible representations. So, for the time being, one may confine attention to either irreducible unitary representations or to appropriate direct integrals of these.

So far nothing has been said about particle concepts related to these states. It is generally assumed that physically interesting systems are capable of a particle interpretation and that all dynamical effects can be described in terms

([15]) See, for example, A. S. Wightman: *Phys. Rev.*, **101**, 860 (1956).
([16]) E. P. Wigner: *Ann. Math.*, **40**, 149 (1939).

of particle interactions and scatterings. All the irreducible unitary representations of the inhomogeneous Lorentz group are known and at present it appears that the only irreducible representations needed are the trivial no-particle (vacuum) representation and the one-particle representations appropriate to particles of zero mass and spin $\frac{1}{2}$ or 1 and finite mass and arbitrary (finite) spin. It is, of course, by no means clear that any interacting field theory could be constructed to give only these states (or can be constructed at all consistently!) within the present formalism.

The vacuum state is invariant under all Lorentz transformations and the one-particle states (belonging to a definite particle type) form an irreducible manifold in the sense that any state can be obtained from any other by means of an appropriate Lorentz transformation. These states are non-degenerate and are « steady » in the sense that a state with a definite value of energy, momentum and helicity belongs only to one such irreducible manifold. The two-particle states on the other hand are neither irreducible nor steady in general; consequent on this fact one can expect scattering processes in two-particle systems.

All these comments are equally true for any many-particle system. Yet there is intuitively a well-defined distinction between two-particle states and all other many-particle states. To make these notions precise we introduce the following characterization. Consider a quantum-mechanical state consisting of two-particles; then all dynamical variables of the system are functionals of the two sets of « particle variables » and the wave function of the system is the product of two one-particle wave functions. This last statement is perhaps made more specific by saying that the (reducible) representation furnished by the two-particle states (which is a direct integral of irreducible one-particle states) is isomorphic to the direct product of two irreducible representations. We now say that a certain state of an *interacting* system belongs to a two-particle manifold if there exists a manifold of states of the interacting system containing the state in question and closed under all Lorentz transformations which is isomorphic to a manifold of states (with the same values of the momentum and helicity) of a non-interacting system containing its two-particle states but to no smaller manifold. (Thus defined a « bound state » of two interacting particles also belongs to a two-particle manifold.) The need for such an elaborate definition is that in a quantum field theory the states are defined in terms of a Hamiltonian operator, etc., as suitable realizations of the Lorentz group but not directly in terms of particle observables. So the particle concepts have to be introduced from outside, and are to some extent arbitrary though restricted by consistency requirements [17]; this point arises

([17]) Compare A. S. WIGHTMAN and S. S. SCHWEBER: *Phys. Rev.*, **98**, 812 (1955); R. ACHARYA and E. C. G. SUDARSHAN: *Journ. Math. Phys.*, **1**, 532 (1960).

even in connection with « one-particle » states for zero mass finite spin cases. It is then clear that the particle interpretation of a field theory is not uniquely defined by the field theory alone but depends on the choice of the construction of particle variables.

We shall now consider the configurational notions associated with a two-particle state. There is first of all the concept of the « distorted » two-particle wave function which is nothing but the expansion coefficients for the states of the two-particle system in terms of the states of the two-particle states of the non-interacting comparison system (chosen so that the continuous energy spectra of the two systems coincide). There is also an intuitive notion of measurement of one-particle properties « when the other particle is far away »; this notion is not sufficiently precise to enable us to proceed with the construction of particle variables. To sharpen this postulate of particle measurement we demand that there exist interactions involving the two-particle states and classical apparatus which converts the two-particle system into a one-particle system; the measurement of one-particle properties can then be made in the standard manner on the states so prepared. The detection of one-particle properties thus consists of a sequence of two operations on the system; the first stage of the compound operation may consist of absorption of the « other » particle (or of a suitably devised « shield »). The important point is that *the other particle may no longer belong to the quantum mechanical state on which the one-particle measurements are performed.* While no non-trivial completely solvable relativistic example is available, for simple non-relativistic models these principles can be illustrated ([18]).

When we consider the more general framework involving the indefinite metric some of these characterizations have to be modified. The no-particle, one-particle, two-particle and many-particle classifications are still valid though the representations furnished for the Lorentz group are not necessarily unitary. However the no-particle and one-particle states do furnish representations; and in general in any representation which is a direct integral of irreducible representations, if one can exhibit a vector of each irreducible manifold which has a definite norm the reducible representation can be used to furnish a unitary representation, in spite of the underlying indefinite metric in the theory. Needless to say there may be manifolds of states which do not furnish a unitary representation; these states can then not be interpreted as physical states. The particle interpretation must then be in accordance with this limitation.

Let us now consider a two-particle manifold of an interacting quantum-mechanical system involving an indefinite metric and consider the scattering

([18]) H. J. Schnitzer and E. C. G. Sudarshan: *Quantum-mechanical systems with indefinite metric* - II, submitted to *Phys. Rev.*

matrix. If there is only one kind of energy eigenstate belonging to the continuous eigenstate, the S-matrix is either unitary (corresponding to real phase shifts) or the states are null vectors. In the first case there is no problem; the second case does not describe a physical system. On the other hand the energy eigenstates may be degenerate and there may be different « kinds » of two-particle states. In this case the S-matrix is in general not diagonal in the label specifying the « composition »; but we may construct the eigenstates of the S-matrix. It is then again clear that only states with normalizable vectors are to be considered; and from the discussion in ref. (3) it is clear that only states with a definite sign of the norm are to be allowed. The totality of all such states (which may be taken to be eigenstates of the S-matrix with positive norm without any loss of generality) appear as candidates for being physical states. It is clear that in the case of a positive definite metric, these conditions imply no restrictions whatever.

There is one more condition to be satisfied by a two-particle state if it is to conform to intuitive notions as defined precisely in an earlier paragraph; namely there should exist the possibility of producing a *physical* one-particle state from this state by removing the « other » particle. This restricts physical states to be the scattering eigenstates with positive definite norm and which map on to two-particle states of the non-interacting system which consists of two physical particles under the defining isomorphism. This is the « asymptotic condition » in our quantum field theory.

One might argue that the « physical » two-particle states so defined are really not pure in composition and contain really different kinds of particles (19). But such a statement implies that it is possible to analyze a given two-particle system and extract from it a negative norm single-particle state; we have seen above that any detection of one-particle properties presupposes an operation of conversion of a two-particle state into a one-particle state. Now all physical operations should, within the postulational framework, connect only physical states. Hence an operation which converts a physical two-particle state into a non-physical one-particle state is not possible and consequently the analysis of the composition of the irreducible physical states is not possible. *This non-analyzability is fundamental to the framework and distinguishes these states from the analyzable eigenamplitudes for scattering of two coupled physical channels.*

The treatment of the external lines thus developed is the following: compute all relevant transition amplitudes in the theory and thus construct the generalized S-matrix of the theory to any desired degree of approximation. No infinities are encountered at any stage of the perturbation series calculation.

(19) The author is indebted to C. J. Goebel for emphasizing that the consistency of the interpretive postulate enunciated in ref. (3) had not been explicitly demonstrated; this demonstration is given in ref. (18).

The S-matrix is now diagonalized and a correspondence is established between these eigenstates of the S-matrix and non-interacting states of a comparison system [18]. The physical states are chosen from amongst those states which are mapped onto many-particle states involving only physical particles. The set of physical states so formed describe physical particles undergoing mutual scattering. An illustration of these ideas for a simple exactly solvable model is given in ref. [18].

In the following section some simple predictions of the theory are mentioned.

5. – Applications.

In the discussion so far we have dealt with general questions and made no specific choices for the masses or specific identifications of the physical particles. We know that there are *two* physical charged leptons namely the electron and the muon, the masses of these particles then fix two of the three independent charged lepton mass parameters. On the other hand for the neutral leptons there is only one neutral lepton known and it has zero mass; so two neutral lepton mass parameters are left undetermined. But the other physical neutral lepton mass must be so high that it does not appear in the decays of pions, kaons or muons, (or it should be practically the same as a neutrino as to be indistinguishable in experiments). This still leaves one parameter undetermined.

We may now consider the decay of the muon in terms of the transition matrix element involving one initial physical particle and three final particles. There are many final states allowed by selection rules but only one physical three-particle state. But considered as a function of the momenta the matrix element would deviate from the predictions of the lowest order calculation by a small amount; we shall not attempt to calculate this correction here but simply mention that the quantitative analysis of these deviations (after the electromagnetic effects have been properly taken into account) should provide some estimates of the undetermined lepton mass parameters [20].

Strictly speaking we cannot discuss any of the observed meson or baryon decays in any quantitative manner since our theory as presented in this paper does not deal with strongly interacting particles. But the qualitative features of the weak interaction process can be displayed by considering a classical source (with a strength given by the vector or axial vector matrix element of the strongly interacting system) to be coupled to the leptons. The extraor-

[20] Such an analysis is now being made by S. HATSUKADE. With regard to the doubling of fermions, compare M. GOLDHABER: *Phys. Rev. Lett.*, **1**, 467 (1958).

dinarily good fit of the predictions of the $V-A$ theory for the $(\pi\rightarrow e+\nu)/(\pi\rightarrow\mu+\nu)$ ratio suggests that the damping of high energy transitions is still unappreciable at this magnitude of the momentum; and this is consistent with the absence of evidence for a neutral counterpart to the muon. On the other hand one expects some damping at the energy appropriate to kaon decay and hence the comparable rates of kaon and pion decays (in spite of the larger phase space in kaon decay) is at least in part due to this. If this is true then it is of great interest to measure precisely the $(K\rightarrow e+\nu)/(K\rightarrow\mu+\nu)$ ratio and compare it with the unique prediction given by the lowest order $V-A$ calculation. The leptonic hyperon decay problem as well as the other leptonic kaon decay modes are considerably more difficult to discuss because of the three-particle final states involved.

6. – Concluding remarks.

In the previous sections we have presented a finite relativistic quantum theory of leptons. The theory as it stands is incomplete since it does not include other particles and interactions, especially the interaction with photons; this will be remedied in subsequent papers of this series. The essential result of this paper is to show that it is possible to construct a consistent theory of four-fermion interactions without having to introduce intermediate vector mesons. The theory also gives a natural *raison d'être* for the muon; and *requires the muon to be degenerate with the electron in its interaction properties.*

The theory does predict two neutral leptons; and in the present state of our experimental knowledge such a prediction is undesirable. But it should be stressed that such a particle is natural in the present formalism; and it is possible that it may have a mass high enough to be not observed in low energy reactions or low enough to be confused for the « usual » neutrino. If the latter circumstance prevails then the measurements usually made would not distinguish this case from the usual assumption of only one kind of neutrino. But it is necessary to stress that since the definition of the « physical particle » in many-particle states involves a certain amount of freedom as explained in Section 4 and in ref. (18), it is possible to arrange the « comparison amplitude » so that the neutrino is the only physical particle which is coupled (21). However this requirement appears somewhat arbitrary and unsymmetric; the question must ultimately be answered only by experiment.

(21) It was mentioned in ref. (3) that in indefinite metric theories a particle interpretation requires the asymptotic condition selecting a subset of positive norm scattering states, (see also Sect. 4) the possibility mentioned in the text corresponds to additional restrictions and implies a further selection among the positive norm steady one-particle states.

The question of the arbitrariness of the details of the present theory invites comment on another aspect, namely the choice of four charged lepton fields and four neutral lepton fields. In earlier studies involving modified propogators, particularly in the outstanding contribution by PAIS and UHLENBECK [22], the several independent fields were brought in by working with Lagrangians involving higher order derivatives. The question naturally arises as to whether our theory can be reformulated in that form; and whether it would be more desirable to start from such a formulation. Straightforward algebraic manipulations enable us to rewrite the theory of *uncoupled* fields in this manner (provided the two « abnormal » fields have equal masses), but the coupling term is no longer simple; this circumstance is not accidental but due to the desire to incorporate the « observed » *universality* of the interaction between *particles* [23]. Nor are the commutation relations of the primitive field of standard type but involve explicitly the self-same masses which we have introduced into our primitive Lagrangian [24]. In view of this it appears that there is no advantage in insisting that the theory be reformulated in terms of a field satisfying higher order equations.

It is also to be pointed out that unlike the motivation of certain earlier speculations involving an additional neutral particle, here there is no attempt to trace a symmetry between the numerical masses of the charged and neutral leptons. Nor is any sanctity attached to the zero mass; anyway, the success of the chiral $V-A$ interaction for the different kinds of four-fermion couplings make any principle directly related to the zero mass of the neutrino somewhat irrelevant.

Similar concepts are certainly applicable to the strongly interacting particles and one might attempt the construction of a theory of fundamental fields, say the Sakata model, as a quantitative theory. But while the framework discussed here gets rid of the infinities the strength of the interaction makes it necessary to study more suitable approximation procedures for these problems than a straight-forward application of perturbation theory. The treatment of strong interactions thus requires additional tools; but these difficulties are not present for the quantum electrodynamics of leptons.

One might argue that the method developed here (or any such involving an indefinite metric and a related interpretive postulate) is *only* a covariant

([22]) A. PAIS and G. E. UHLENBECK: *Phys. Rev.*, **79**, 145 (1950).

([23]) This is to be contrasted with the situation arising from coupling the primitive field (obeying the higher order equation) *directly*. In this latter case the particles will not be coupled universally; compare eq. (54), (57) and (58) of ref. ([22]).

([24]) The « unaesthetic arbitrariness » of this Lagrangian (as well as of the Lagrangians of current theories!) has been strongly criticized by I. BIALYNICKI-BIRULA in discussions with the author.

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form-factor [25]. Such an argument would certainly be valid; in fact, as argued in an earlier paper, the use of the indefinite metric is only as an aid to simplicity in constructing a finite relativistic quantum theory.

* * *

The author is indebted to I. BIALYNICKI-BIRULA, P. CZIFFRA, B. P. NIGAM and H. SCHNITZER for critical comments.

APPENDIX

Renormalization.

We remarked in Sect. 3 that while the perturbation theory developed there contains no infinite terms, there is still a difference between the « physical » mass of the various leptons and the « bare » mass parameter occurring in the original equation; similarly there will be higher order corrections to the various four-fermion reactions. The renormalization program for the present theory is essentially the same as in the case of the usual « renormalizable theories » except that in the present theory we have only finite renormalizations to carry out. The discussion below follows the pattern of JAUCH and ROHRLICH [13].

Consider an arbitrary diagram; it is made up of an arbitrary number of charged lepton lines, neutral lepton lines and four-fermion vertices; there are 8 kinds of lepton lines and 256 kinds of vertices. According to the choice of the universal $V-A$ four-fermion interaction all these vertices correspond to the same coupling constant and coupling type. We define a « self-energy part » as any part of a diagram which is connected to the rest of the diagram by exactly two lines. In the present theory it is necessary that both the lines are either charged lepton lines or neutral lepton lines, but the two lines need not belong to the same type of leptons; consequently the general self-energy part is a second rank tensor in the indices j, j' referring to the two lines in which the self-energy part is terminated. Similarly we define a « vertex part » as any part of a diagram which is connected to the rest of the diagram by exactly two charged lepton lines and two neutral lepton lines. The vertex part so defined is a tensor of the fourth rank in the two sets of two indices. (Note that the four-legged parts involving all charged or all neutral lines is *not* a vertex part according to this definition.) To every diagram there corresponds a « skeleton diagram » obtained by replacing all inserted self-energy

(25) Note that the form-factor involved here is a *dynamical* form-factor and manifests itself differently in different states; the question whether a theory involving a non-dynamical form-factor can be consistent remains open.

parts by lepton lines [26] and all inserted vertex parts by simple four-fermion vertices. A diagram identical with its own skeleton is called an « irreducible » diagram. As in the usual case we see that the only irreducible self-energy diagram is the second order diagram (Fig. 3) (though there are such self-energy parts for the eight leptons), since any self-energy diagram starts with one four-fermion vertex and the remaining part of the diagram is a vertex part according to the above definition: hence in any skeleton diagram the remaining part is also a simple four-fermion vertex. But there are an infinite number of irreducible vertex parts.

Let $\Sigma_\psi^{jj'}(p)$ and $\Sigma_\chi^{jj'}(p)$ refer to the charged and neutral lepton self-energy tensors summed over all diagrams and let $S_\psi^{jj'}(p) = S_\psi(p;j)\delta_{jj'}$ and $S_\chi^{jj'}(p) = S_\chi(p;j)\delta_{jj'}$ be the bare propagation functions. The modified propagation functions $S'^{jj'}(p)$ are then given by

$$\text{(A.1)}\qquad \left\{\begin{aligned} S_\psi'^{jj'}(p) &= S_\psi(p;j)\delta_{jj'} + S_\psi(p)\Sigma_\psi^{jj'}(p)S_\psi(p)\,,\\ S_\chi'^{jj'}(p) &= S_\chi(p;j)\delta_{jj'} + S_\chi(p)\Sigma_\psi^{jj'}(p)S_\chi(p)\,.\end{aligned}\right.$$

And the modified vertex part is given by

$$\text{(A.2)}\qquad \Gamma^{j_1j_2j_3j_4}(p_1, p_2, p_1', p_2') = \gamma_\mu(1+\gamma_5)\times\gamma^\mu(1+\gamma_5) + \Lambda^{j_1j_2j_3j_4}(p_1, p_2, p_1', p_2')\,,$$

where the first term on the right-hand side corresponds to the primitive chiral $V-A$ coupling and the second term represents the sum over all *proper* vertex parts, a « proper » diagram being defined as one which cannot be separated into two disjoint diagrams by opening a single line: the contributions from improper diagrams have already been included in the modification of the propagation functions. In view of the chiral $V-A$ interaction under recouplings the order of coupling the four fields in (A.2) is irrelevant.

We now separate the contribution from all the diagrams into « renormalization » and « physical » parts. For this purpose we rewrite the inertia term in (8) in the form

$$\text{(A.3)}\qquad \sum_{j,j'} M^{jj'}\overline{\psi}^{(j)}\psi^{(j')} = \sum_j m^{(j)}\overline{\psi}^{(j)}\psi^{(j)} = \sum_{r=1}^{4} m_0^{(r)}\overline{\psi}_{(r)}\psi_{(r)} - \sum_{j,j'} A^{jj'}\overline{\psi}^{(j)}\psi^{(j')}\,,$$

where the modified field $\psi_{(r)}$ is given by

$$\text{(A.4)}\qquad \psi_{(r)} = \sum_j v_\psi^j(r)\,\psi^{(j)}\,.$$

The unitary matrix $v^j(r)$ and the numerical matrix $A^{jj'}$ are as yet undetermined and are to be determined as follows. Using the new masses $m_0^{(r)}$ and the new fields $\psi_{(r)}$ we may define the propagators $S_\psi^{(r)}(p)$. Let us first consider

[26] There is a slight difference in this operation from the usual case because of the tensor character od the self-energy parts; but no troubles arise from omitting an explicit recognition of this in our work.

the irreducible self-energy parts (Fig. 3) which are of the second order. We note that according to (A.1) the self-energy part is a tensor in the indices j, j' and may be written in the form

$$\text{(A.5)}\qquad \Sigma_{\psi}^{jj'}(p) = A_{\psi}^{jj'} - \sum_{r=1}^{4} B_{\psi}^{(r)} S_{\psi}^{-1(r)}(p)\, v_{\psi}^{j*}(r)\, v_{\psi}^{j'}(r)\, + \sum_{r,r'=1}^{4} S_{\psi}^{-1(r)}(p)\, M^{(rr')}(p)\, S_{\psi}^{-1(r')}\, v_{\psi}^{j*}(r)\, v_{\psi}^{j'}(r')\,,$$

and we require this to be computed using the new propagators $S_{\psi}^{(r)}(p)$. A completely analogous expression can be obtained for the neutral lepton self-energy part. Similarly we write the vertex modification by all the irreducible contributions as the sum of the renormalization and physical parts:

$$\text{(A.6)}\qquad \Lambda'^{\,j_1j_2j_3j_4}(p_1, p_2, p_1', p_2') = L^{j_1j_2j_3j_4}\gamma_{\mu}(1+\gamma_5)\times\gamma^{\mu}(1+\gamma_5) + \sum_{r_1r_2r_3r_4} \Lambda^{(r_1r_2r_1'r_2')}(p_1, p_2, p_1', p_2')\, v_{\psi}^{j_1*}(r_1)\, v_{\chi}^{j_2}(r_2)\, v_{\psi}^{j_3}(r_1')\, v_{\chi}^{j_4*}(r_2')\,.$$

In (A.5) and (A.6) the separation into the two parts is to be made definite by the requirement that for all the momenta p_1, p_2, p_1', p_2' on the mass-shell for the modified « particles » denoted by r_1, r_2, r_1', r_2' and the primed and unprimed quantities being taken equal and at the threshold for the reaction $p_1 + p_2 \rightarrow p_1' + p_2'$ the physical correction part of (A.6) vanishes.

We now proceed to the separation of the physical parts of proper reducible diagrams by a method of induction. For this purpose we take any diagram of arbitrary order and assume that diagrams of all lower orders have been separated into the renormalization and physical parts. We now replace each vertex part and each propagator by the sum of the unmodified part plus the physical part *i.e.*, drop all renormalization parts of the modifications; separate the physical part of the proper diagram so computed and sum the contributions so obtained from all the proper (reducible and irreducible) diagrams to obtain expressions formally identical with (A.1) and (A.2). These equations indicate a shift in mass of the modified fermion « r ». To define « physical » particles we now determine $v_{\psi}^{j}(r)$ so that by treating the term $m_0^{(r)}\overline{\psi}_{(r)}\psi_{(r)}$ as the inertia term in the calculation both (A.3) and (A.5) can be consistent with each other. When such a choice is made the mass renormalization is complete and in all further calculations the mass shift of the (modified) « free » particles may be taken to vanish. It is important to note that in the present theory the « mass renormalization » not only redefines the mass of the physical particles but also defines the physical particles in terms of the modified fields.

Let us now complete the renormalization by introducing the physical propagators, vertex functions and renormalized coupling constants. Define

$$\text{(A.7)}\qquad S_{0\psi}^{-1(r,r')}(p) = S_{\psi}^{-1(r)}(p)\delta_{rr'} + M_{\varphi}^{(rr')}(p)\,,$$

$$\text{(A.8)}\qquad S_{0\chi}^{-1(r,r')}(p) = S_{\chi}^{-1(r)}(p)\delta_{rr'} + M_{\chi}^{(rr')}(p)\,,$$

$$\text{(A.9)}\qquad \Gamma_0^{(r_1r_2r_1'r_2')}(p_1p_2p_1'p_2') = \gamma_{\mu}(1+\gamma_5)\times\gamma^{\mu}(1+\gamma_5) + \Lambda^{(r_1r_2r_1'r_2')}(p_1p_2p_1'p_2')\,.$$

We now assert that

$$\text{(A.10)} \qquad \left\{ \begin{aligned} S_{0\psi}^{(jj')}(\{G_0\}) &= \{Z_\psi^{(j)} Z_\psi^{(j')}\}^{-\frac{1}{2}} S_\psi^{\prime jj'}(\{G\}) \,, \\ S_{0\chi}^{(jj')}(\{G_0\}) &= \{Z_\chi^{(j)} Z_\chi^{(j')}\}^{-\frac{1}{2}} S_\chi^{\prime jj'}(\{G\}) \,, \end{aligned} \right.$$

$$\text{(A.11)} \qquad \Gamma_0^{(j_1 j_2 j_1' j_2')}(\{G_0\}) = \{Y_\psi^{(j_1)} Y_\chi^{(j_2)} Y_\psi^{(j_1')} Y_\chi^{(j_2')}\}^{+\frac{1}{2}} \Gamma^{j_1 j_2 j_1' j} (\{G\}) \,.$$

with $\{G_0\}$ denoting a new set of 256 renormalized coupling constants:

$$\text{(A.12)} \qquad G_0(j_1 j_2 j_1' j_2') = \sqrt{\frac{Z_\psi^{(j_1)} Z_\chi^{(j_2)} Z_\psi^{(j_1')} Z_\chi^{(j_2')}}{Y^{(j_1 j_2 j_1' j_2')}}} \cdot G \,.$$

Along with the propagator renormalization there is also a wave function renormalization. The proof of the assertions embodied in (A.10) to (A.12) consists in showing that matrix elements computed in the usual fashion from the original Feynman graphs are identical with the result of computing only the physical part but with the renormalized coupling constants (A.12), provided the mass renormalization (including the definition of the physical combinations) is already carried out and provided that the renormalization constants are defined by the equations:

$$\text{(A.13)} \qquad Z^{(j)} = 1 - B^{(j)}(\{G_0\}) \,,$$

$$\text{(A.14)} \qquad Y^{(j_1 j_2 j_1' j_2')} = 1 - L^{(j_1 j_2 j_1' j_2')}(\{G_0\}) \,.$$

This may be verified in a straightforward fashion.

The renormalization constants (A.13) and (A.14) are thus given as power series in the renormalized coupling constants. The question as to whether these quantities are finite or not would then depend upon the convergence of these power series. There is no reason why the series should converge, though in the present treatment the coefficients of the power series are all finite quantities and an estimate of the expansion parameter made in the previous section shows it to be very small. Further, the study of certain simple models exhibits the analyticity of the functions (of the coupling constants) defined by the power series ([18]). In the absence of any proof to the contrary we take the renormalization constants to be finite and the theory to be well-defined.

RIASSUNTO (*)

Formulo una teoria relativistica finita delle interazioni dei quadrifermioni; la teoria comporta come componente essenziale l'uso di una metrica indefinita. I problemi di interpretazione che sorgono con l'uso di una metrica indefinita vengono analizzati in relazione agli osservabili ed alla struttura degli stati a molte particelle nella teoria quantistica dei campi; e si dimostra la coerenza dei postulati interpretativi. La teoria fornisce incidentalmente una « ragion d'essere » per il muone.

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Some Partition Problems with Analogies in Quantum Statistics.

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Summary. — The problem of counting configurations of spins in Ising lattices which satisfy various algebraic conditions is formally similar to computing the entropy of certain many-particle systems obeying Fermi-Dirac statistics. Steepest descent methods are used to solve several of these problems, and the analogies with quantum statistical formulae noted.

1. – Introduction.

Some counting problems have been encountered in a new approach to the 3-dimensional Ising lattice which can be solved (at least in the interesting asymptotic ranges) by techniques already known in quantum statistical mechanics. That is, solving these purely mathematical partition problems may be interpreted as finding the entropy of certain statistical assemblies under various « conditions » (constraints). It is noteworthy that the rules of the counting demand quantum, rather than classical, statistics for the states of the formal many-particle assemblies so defined. For example, when one is counting different configurations of spins distributed over the Ising lattice which satisfy certain algebraic conditions, the situation in which lattice sites i and j are « occupied by spins-up » gives naturally a single configuration. One must not try (and there is no temptation) to distinguish the two spins-up and count as different configurations the situation: « spin-up 1 occupies i; spin-up 2 occupies j » and the situation: « spin-up 1 occupies j; spin-up 2 occupies i. » In the hope that this analogy may prove useful in wider contexts than the particular problems mentioned here, we publish the following note.

2. – A preliminary counting problem.

As a preliminary, simpler problem, say that one defines

$$a^+(\boldsymbol{r}_1^+, \boldsymbol{r}_2^+, \ldots \boldsymbol{r}_{N^+}^+; \boldsymbol{k}) \equiv \sum_{i=1}^{N^+} \cos \boldsymbol{k}\cdot\boldsymbol{r}_i^+ , \tag{2.1}$$

where N^+ spins-up are distributed over the lattice of $\mathcal{N} \geqslant N^+$ points [1] at the points $\boldsymbol{r}_i^+$, $i = 1 \ldots N^+$. The other Ising lattice points then have spin down, of course. $\boldsymbol{k}$ is a certain wave number whose definition is unimportant here. The question is: in how many ways can this be done such that $a^+(\boldsymbol{r}_1^+, \boldsymbol{r}_2^+, \ldots \boldsymbol{r}_{N^+}^+; \boldsymbol{k}) =$ = some fixed number a? We can rephrase it this way: consider the equation

$$a = \sum_{i=1}^{\mathcal{N}} n_i \varepsilon_i , \qquad \varepsilon_i = \cos \boldsymbol{k}\cdot\boldsymbol{r}_i \tag{2.2}$$

with

$$n_i = \begin{cases} 1 & \text{if lattice site } \boldsymbol{r}_i \text{ has spin up}, \\ 0 & \text{if lattice site } \boldsymbol{r}_i \text{ has spin down}. \end{cases} \tag{2.3}$$

Then we ask: how many *configurations* $\{n_i\}$ satisfying (2.2) exist? This is the same as asking for the antilogarithm of the entropy of a Fermi-Dirac many-particle system with single-particle « energy levels » $\varepsilon_i = \cos \boldsymbol{k}\cdot\boldsymbol{r}_i$ in the micro-canonical ensemble of total « energy » $E = a$, with no restriction on the number of particles participating (except of course the inequalities $0 \leqslant \sum n_i \leqslant \mathcal{N}$). This problem, for $\mathcal{N}$ very large and large E, can be solved by a steepest descent method which is a slight modification of the treatment given, say in Schrödinger's book [2] or in the book [3] of Fowler and Guggenheim. We state the problem in a slightly more general form, then write down the answer.

Consider the equation

$$E = \sum_{i=1}^{\mathcal{N}} n_i \varepsilon_i , \qquad n_i = 0 \text{ or } 1. \tag{2.4}$$

The restrictions on the single particle spectrum are

(1) As is customary in statistical mechanics N^+ and N (Sect. 4) are reserved for « numbers of particles ». The script symbol $\mathcal{N}$ refers to the number of single-particle states, which happens to be finite in these formal Fermi-Dirac systems.

(2) E. Schrödinger: *Statistical Thermodynamics* (Cambridge, 1946).

(3) R. Fowler and E. Guggenheim: *Statistical Thermodynamics* (Cambridge, 1952).

a) There are $\mathcal{N}$ ($\mathcal{N}$ very large) single particle states, labelled by i, of energies ε_i. The ε_i are integers whose g.c.d. = unity. (This must be attainable by a suitable choice of the energy unit.)

b) Positive, negative, and zero ε_i are allowed, but the number of either positive or negative ε_i is large, say $= O(\mathcal{N})$. (For definiteness, in this paper we assume, that the number of positive ε_i is large, $= O(\mathcal{N})$.

c) When $\mathcal{N}$ is made to approach ∞ by varying external parameters (volume of a gas container, edge of a lattice with fixed lattice constant, etc.), the energy density of single particle states has the form $w(\varepsilon) = \mathcal{N}\theta(\varepsilon)$, $\theta(\varepsilon)$ independent of $\mathcal{N}$, for large $\mathcal{N}$.

Conditions *a*), *b*), *c*) are probably sufficient that steepest descent methods can be applied and yield asymptotically exact answers. Then if $\exp[\sigma(E)] =$ = number of configurations satisfying (2.4) ($\sigma(E)$, the entropy of the system in the condition (2.4))

$$\sigma(E) = -E\log\zeta + \sum_i \log(1+\zeta^{\varepsilon_i}) - \frac{1}{2}\log\left[\frac{\pi}{2}\frac{1}{\zeta^2}\sum_i \varepsilon_i^2\,\mathrm{sech}^2\left(\frac{\varepsilon_i}{2}\log\zeta\right)\right], \tag{2.5}$$

where the parameter $\zeta = \zeta(E)$ is given implicitly by

$$E = \sum_i \frac{\varepsilon_i}{1+\zeta^{-\varepsilon_i}}, \tag{2.6}$$

E is supposed large, and a 1 has been neglected relative to E in (2.5), (2.6). The last term in (2.5) is of lower order than the first two owing to assumption *c*) and may often be neglected.

For the particular problem given at the beginning of this section

$$w(\varepsilon) = \frac{\mathcal{N}}{\pi}\frac{1}{(1-\varepsilon^2)^{\frac{1}{2}}}, \qquad \text{(suitable } \boldsymbol{k}\text{)}, \tag{2.7}$$

and changing sums to integrals, (2.6) takes either of the forms

$$\frac{\pi}{2}\frac{E}{\mathcal{N}} = \int_0^{\pi} \frac{\cos\varphi}{1+\zeta^{-\cos\varphi}}\,d\varphi = \int_{-\pi/2}^{\pi/2} \frac{\sin\theta\,d\theta}{1+\exp[-\sin\theta\log\zeta]}. \tag{2.8}$$

The second integral shows clearly that it is an odd function of $\log\zeta$.

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$\pi E/2\mathcal{N}$ is plotted as a function of $\log \zeta$ in Fig. 1; it is not expressible in terms of known functions.

The theory applies here even though the ε_i of (2.2) are not integers. The reason is that the (purely combinatorial) result (2.5) cannot depend on the choice of the energy unit, which can thus be taken so small that the spectrum (2.2) approximates an « integral » one as closely as desired. The formal proof that (2.5) *cum* (2.6) is independent of the energy unit follows immediately from the observation that if the ε_i and E are integers (the actual energies in terms of the energy unit ε) and the larger integers $\varepsilon_i' = \alpha\varepsilon_i$, $E' = \alpha E$, $\alpha =$ positive integer > 1, relative to the smaller energy unit $\varepsilon' = \varepsilon/\alpha$, then

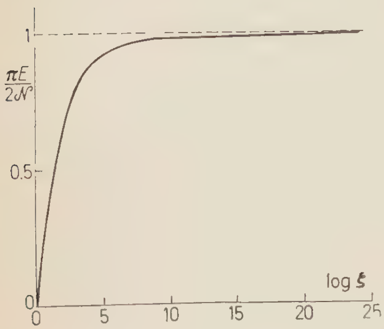

Fig. 1. – The parameter ζ as function of E for the spectrum $\varepsilon_i = \cos \boldsymbol{k}\cdot\boldsymbol{r}_i$, $i = 1, 2, \ldots \mathcal{N}$.

$$\zeta' = \zeta^{1/\alpha} \tag{2.9}$$

from (2.6).

Note also that ζ is a function only of the *intensive* $E/\mathcal{N}$ (cf. (2.8)), therefore if $\mathcal{N} \to \infty$ and $E \to \infty$ such that $E/\mathcal{N} =$ const., the first terms in (2.5) are $O(\mathcal{N})$, the last term only $O(\log \mathcal{N})$.

3. – Complex spectra.

The problem actually encountered in the work on the Ising lattice was to find in how many ways one could distribute spins-up over the lattice such that the sum

$$\sum_{i=1}^{N^+} \exp\left[-i\,\boldsymbol{k}\cdot\boldsymbol{r}_i^+\right]$$

equals some fixed complex number A. Unfortunately, the theory of Section 2 does not work for complex energy levels, so that the problem becomes one of finding the entropy of a system in a condition specified by two constraints. Similar problems for real quantum systems have been solved [4] but always with the special spectra dictated by the physics. We therefore generalize those techniques slightly.

[4] See, for example, reference [3], § 215.

Consider the simultaneous equations

$$E = \sum_{i=1}^{\mathcal{N}} n_i \varepsilon_i\,, \qquad F = \sum_{i=1}^{\mathcal{N}} n_i \delta_i\,; \qquad n_i = 0 \text{ or } 1, \tag{3.1}$$

where $\{\varepsilon_i\}$ and $\{\delta_i\}$ are single particle spectra of the type described by assumptions a), b), c). We ask for the number $\exp[\sigma(E, F)]$ of solutions $\{n_i\}$ satisfying (3.1). $\sigma(E, F)$ is the entropy of the formal FD system in the condition specified by (3.1). Define

$$f(z, w) \equiv \prod_i (1 + z^{\varepsilon_i} w^{\delta_i}) \tag{3.2}$$

and

$$\exp[g(z, w)] \equiv z^{-E-1} w^{-F-1} f(z, w)\,, \tag{3.3}$$

where z and w are independent complex variables. Then the desired number is given by a double contour integral:

$$\exp[\sigma(E, F)] = \left(\frac{1}{2\pi i}\right)^2 \oint\oint \mathrm{d}z\,\mathrm{d}w \exp[g(z, w)]\,, \tag{3.4}$$

where the contours encircle the respective origins, etc.

For general ε_i and δ_i this problem is not exactly identical with any found in quantum statistics. For all $\delta_i = 1$ it could be interpreted as finding the entropy in the microcanonical ensemble E with fixed number F of fermions.

When (3.4) is evaluated by the method of steepest descent one gets

$$\sigma(E, F) = -E\log\zeta - F\log\omega + \sum_i \log(1 + \zeta^{\varepsilon_i}\omega^{\delta_i}) - \\ - \frac{1}{2}\log\left[\left(\frac{\pi}{2}\right)^2 \frac{1}{\zeta^2\omega^2}\left\{\|\varepsilon\|^2\|\delta\|^2 - (\varepsilon, \delta)^2\right\}\right], \tag{3.5}$$

where the parameters $\zeta(E, F)$ and $\omega(E, F)$ are determined by

$$E = \sum_i \frac{\varepsilon_i}{1 + \zeta^{-\varepsilon_i}\omega^{-\delta_i}}\,, \qquad F = \sum_i \frac{\delta_i}{1 + \zeta^{-\varepsilon_i}\omega^{-\delta_i}}\,. \tag{3.6}$$

The last term in (3.5), which is again only $O(\log \mathcal{N})$ and may often be dropped,

involves the norms and inner products [5] defined by

$$(3.7) \qquad (\varepsilon, \delta) \equiv \sum_i \varepsilon_i \delta_i g_i \,, \qquad \|\varepsilon\|^2 \equiv (\varepsilon, \varepsilon) \,,$$

$$g_i \equiv \operatorname{sech}^2 [\tfrac{1}{2}(\varepsilon_i \log \zeta + \delta_i \log \omega)] \,.$$

The particular problem cited at the beginning of this section is the case

$$(3.8) \qquad \begin{cases} \varepsilon_i = \cos \boldsymbol{k} \cdot \boldsymbol{r}_i \,, & \delta_i = -\sin \boldsymbol{k} \cdot \boldsymbol{r}_i \,, \\ E = \operatorname{Re} A \,, & F = \operatorname{Im} A \,. \end{cases}$$

4. – Meaning of the parameter ζ.

The average occupation number of the i-th level, $\bar{n}_i$, defined as the average over the configurations satisfying the single condition (2.4), can be likewise computed by the method of steepest descent [6] and comes out to be

$$(4.1) \qquad \bar{n}_i = \frac{1}{1 + \zeta^{-\varepsilon_i}} \,.$$

Then our $\sigma(E)$ must agree (at least in lowest order) with the thermodynamical entropy $S(T)/k$ for that particular $\zeta(T)$ such that $E(\zeta)$, (2.6), is the thermodynamical energy

$$(4.2) \qquad E(T) = \sum_i \frac{\varepsilon_i}{1 + \exp[(\varepsilon_i - \mu)/kT]} \,.$$

This gives $\zeta(T) = \exp[-1/kT]$ if the chemical potential $\mu = 0$, *i.e.*, if the system is one in which the number of particles is not fixed but adjusts itself to the temperature in accordance with

$$(4.3) \qquad N(T) = \sum_i \bar{n}_i \,, \qquad (\zeta(T) = \exp[-1/kT]) \,.$$

Putting these into (2.5), we get

$$(4.4) \qquad \sigma[E(T)] = \sum_i \left\{ \frac{\varepsilon_i/kT}{1 + \exp[\varepsilon_i/kT]} + \log(1 + \exp[-\varepsilon_i/kT]) \right\} + O(\log \mathcal{N}) \,,$$

[5] One advantage of using these Hilbert-space notations is that the argument of the log in the last term of (3.5) is immediately seen to be non negative. In fact, it can vanish only in the case that $\varepsilon_i \propto \delta_i (i = 1, \dots \mathcal{N})$, which case is excluded anyway, since then the problem degenerates into the simpler problem of Section 2.

[6] See, for example, reference [3].

which can be rearranged to give the well-known expression

$$\sigma[E(T)] = S(T)/k = -\sum_i \{\bar{n}_i \log \bar{n}_i + (1 - \bar{n}_i) \log (1 - \bar{n}_i)\} + O(\log . V) \tag{4.5}$$

in terms of the $\bar{n}_i$, (4.1), *q.e.d.*

* * *

Thanks are due to Dr. O. THEIMER and Mr. R. GENTRY for stimulating discussions.

RIASSUNTO (*)

Il problema del conteggio delle configurazioni di spin nei reticoli di Ising, che soddisfino varie condizioni algebriche, è formalmente simile al calcolo dell'entropia di alcuni sistemi a molte particelle che seguono la statistica di Fermi-Dirac. Uso metodi di deduzione rapida per risolvere molti di questi problemi, e faccio notare le analogie con le formule della statistica quantistica.

(*) *Traduzione a cura della Redazione.*

Two-Particle Structure of the Mandelstam Representation.

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(ricevuto l'11 Marzo 1961)

Summary. — The two-particle contributions to the Mandelstam representation of the scattering amplitude are separated for the three reaction channels simultaneously.

Introduction.

Recently WILSON [1,2] has given a clear formulation of the crossing symmetric two particle approximation of the scattering amplitude which was indicated by MANDELSTAM's work [3,4]. In this approximation the meson-meson scattering amplitude satisfies the Mandelstam representation and the unitarity condition in the elastic region of each reaction channel. In the inelastic regions approximations are made for the unitarity condition, but in such a manner that the crossing symmetry is not violated.

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(1) K. G. WILSON: preprint (1960), to be published.

(2) The same results have been found independently by TER-MARTIROSIAN: *Žurn. Èksp. Teor. Fiz.*, **39**, 827 (1960). The author is grateful to Dr. LASCOUX for informing him about this work.

(3) S. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958); **115**, 1741, 1752 (1959), in the following quoted as M1, M2 and M3. For the meson-nucleon scattering a relativistic and crossing symmetric one-meson approximation was formulated in M1 in terms of perturbation theory.

(4) From a quite different point of view the same approximation scheme has been proposed by CHEW and FRAUTSCHI. In this work particular emphasis is laid on a unified tratment of high energy and low energy phenomena. For a preliminary report see G. F. CHEW and S. C. FRAUTSCHI: *Phys. Rev. Lett.*, **5**, 580 (1960).

As will be shown in this paper, WILSON's work can be easily generalized to a crossing symmetric treatment of the two particle structure of the scattering amplitude provided that the Mandelstam representation is satisfied. In carrying out this program we shall frequently make use of the convenient technique which has been developed by MANDELSTAM for handling the elastic unitarity condition and the analytic properties of the scattering amplitude.

The analogous problem for the two particle propagation function has been solved by SYMANZIK (5) on the basis of a crossing symmetric Bethe-Salpeter type equation. This result is significant in so far as it can be obtained without using the conjecture of the Mandelstam representation. If applied to the scattering amplitude, Symanzik's formalism also leads to a crossing symmetric treatment of the two particle structure, however, in terms still involving the propagation function off the mass shell.

Some well known consequences of the Mandelstam representation are collected in Sections **1** and **2**. Using these results it will be proved (Section **3**) that the scattering amplitude T can be decomposed in the form

$$T = T_{\text{el}} + T_{\text{inel}} ,$$

where the elastic part T_{el} and the inelastic part T_{inel} again satisfy a Mandelstam representation with full crossing symmetry. The spectral function ϱ_{inel} of the inelastic part T_{inel} vanishes in the elastic region of each reaction channel. Consequently T_{inel} has the same symmetry and analytic properties as T except that the three cuts start at the first inelastic threshold. In Section **4** two coupled non linear integral equations (1-4) are derived which determine the scattering amplitude for given inelastic spectral functions ϱ_{inel}. By any solution of these equations an amplitude can be constructed satisfying crossing symmetry, spectral representation and the unitarity condition in the elastic region (6). If the inelastic part is neglected these integral equations represent Mandelstam's two particle approximation (1-4).

In Section **5** the method is extended to some other scattering processes. In particular the meson-nucleon scattering (spins disregarded) is discussed. As the treatment of the nucleon-antinucleon annihilation encounters certain difficulties only the one meson structure of the meson-nucleon channels will be studied. Neglect of the inelastic spectral functions then leads to Mandelstam's crossing symmetric and relativistic one meson approximation (M1).

(5) K. SYMANZIK: *Report*, Rochester Conference 1960 and private communications. The same equations have been obtained independently by T. T. WU on the basis of perturbation theory (private communication).

(6) Unitary scattering amplitudes can be constructed in many ways, see for instance R. BLANKENBECLER: preprint (1960). In general, however, the amplitudes constructed do not have the full analytic properties of the Mandelstam representation.

1. – Analytic properties.

We consider the scattering of two identical particles of mass m, charge and spin zero. Let k_1, k_2 denote the momenta of the incoming, k_1', k_2' the momenta of the outgoing particles. For the scattering amplitude T we use the notation

$$(\Phi^{\text{out}}_{k_1k_2}, \Phi^{\text{in}}_{k_1k_2}) - (\Phi^{\text{i}}_{k_1'k_2'}, \Phi^{\text{in}}_{k_1k_2}) = i\,T(k_1k_2|k_1'k_2') = i\,\delta(k_1' + k_2' - k_1 - k_2)\,T(s, t)\,,$$

with

$$s = -(k_1 + k_2)^2\,, \qquad t = -(k_1 - k_1')^2\,.$$

For some causal and relativistic interactions it has been shown in some order of perturbation theory that the analytical continuation $\Phi(s, t)$ of the scattering amplitude $T(s, t)$ satisfies the Mandelstam representation [7]. In this paper we will assume the Mandelstam representation and discuss the unsubtracted form [8]

$$\begin{aligned}(1)\qquad \Phi(s, t) = \lambda &+ \frac{1}{\pi}\int \mathrm{d}s'\,\frac{\varrho(s')}{s'-s} + \frac{1}{\pi}\int \mathrm{d}t'\,\frac{\varrho(t')}{t'-t} + \frac{1}{\pi}\int \mathrm{d}u'\,\frac{\varrho(u')}{u'-u} + \\ &+ \frac{1}{\pi^2}\int \mathrm{d}s'\,\mathrm{d}t'\,\frac{\varrho(s', t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int \mathrm{d}t'\,\mathrm{d}u'\,\frac{\varrho(t', u')}{(t'-t)(u'-u)} + \\ &+ \frac{1}{\pi^2}\int \mathrm{d}u'\,\mathrm{d}s'\,\frac{\varrho(u', s')}{(u'-u)(s'-s)}\,,\end{aligned}$$

where $u = 4m^2 - s - t$. The boundary values along the cut $s \geqslant 4m^2$ are

$$T(s, t) = \lim_{\varepsilon \to +0} \Phi(s + i\varepsilon, t) \qquad \text{for} \quad s \geqslant 4m^2\,, \quad 0 \leqslant -t \leqslant s - 4m^2\,,$$

$$T^*(s, t) = \lim_{\varepsilon \to +0} \Phi(s - i\varepsilon, t) \qquad \text{for} \quad s \geqslant 4m^2\,, \quad 0 \leqslant -t \leqslant s - 4m^2\,.$$

The spectral functions $\varrho(s)$ and $\varrho(s, t)$ are real and symmetric

$$(2) \qquad \varrho(s, t) = \varrho(t, s)\,.$$

(7) M2 (in fourth order); G. Wanders: to be published (in sixth order). The Mandelstam representation has been discussed in any order of perturbation theory, but the proofs given so far are incomplete. T. T. Wu and C. N. Yang: private communication; H. Enz and J. Lascoux: private communication; R. J. Eden, P. V. Landshoff, J. G. Polkinghorne and T. C. Taylor: preprint 1961.

(8) If the integrals over the ϱ diverge (1) must be replaced by subtracted forms (M1, 2, and ref. (1)). The necessary modifications of the relations given in this paper can be easily carried out.

The relations of crossing symmetry are

(3) $$\Phi(s, t) = \Phi(t, s) = \Phi(s, 4m^2 - s - t) = \Phi(4m^2 - s - t, t)\,,$$

i.e. Φ is fully symmetric in s, t, and u.

For simplicity we consider a pair theory ([9]) so that the three particle vertex function vanishes identically (the general case with non vanishing vertex function is discussed in Section 5). We further exclude stable bound states and any coupling to particles of a different kind. Then no discrete masses occur in the discussions of (1) and we have

(4) $$\varrho(s) = \varrho(s, t) = 0 \quad \text{if} \quad s < 4m^2\,, \qquad t < 4m^2\,.$$

The discontinuity across the s cut will be denoted by

(5) $$2iA(s, t) = \lim_{\varepsilon \to +0} \big(\Phi(s + i\varepsilon, t) - \Phi(s - i\varepsilon, t)\big) \qquad \text{for } s \geqslant 4m^2\,,$$

and we will use the convention $A(s, t) = 0$ for all $s < 4m^2$.

By crossing symmetry (3) the discontinuities across the other cuts are

$$\lim_{\varepsilon \to +0} \big(\Phi(s, t + i\varepsilon) - \Phi(s, t - i\varepsilon)\big) = 2iA(s, t) \qquad \text{for } t \geqslant 4m^2\,,$$

$$\lim_{\varepsilon \to +0} \big\{\big(\Phi(s, 4m^2 - s - u - i\varepsilon) - \Phi(s, 4m^2 - s - u + i\varepsilon)\big)\big\} = 2iA(u, s)\,, \qquad u \geqslant 4m^2.$$

For a fixed energy the scattering amplitude T and the absorptive part A satisfy a dispersion relation in t

(6) $$T(s, t) = C(s) + \frac{1}{\pi}\int \mathrm{d}t' \,\frac{A(t', s + i\varepsilon)}{t' - t} + \frac{1}{\pi}\int \mathrm{d}u' \,\frac{A(u', s + i\varepsilon)}{u' - u}\,, \qquad \varepsilon \to +0\,,$$

(7) $$A(s, t) = \varrho(s) + \frac{1}{\pi}\int \mathrm{d}t' \,\frac{\varrho(s, t')}{t' - t} + \frac{1}{\pi}\int \mathrm{d}u' \,\frac{\varrho(s, u')}{u' - u}\,.$$

The constant term $C(s)$ in (6) is given by

(8) $$C(s) = \lambda + \frac{1}{\pi}\int \mathrm{d}s' \,\frac{\varrho(s')}{s' - s - i\varepsilon}\,, \qquad \varepsilon \to +0\,.$$

([9]) *I.e.*, the theory is invariant under the transformation

$$A(x) \to -A(x)$$

of the field operator $A(x)$.

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For $s > 4m^2$ the scattering amplitude and the absorptive part are analytic functions of

$$\cos\theta = 1 + \frac{2t}{s - 4m^2},$$

regular on the complex $\cos\theta$-plane except for cuts along the real axis. With the notation

$$T(s;\, \cos\theta) = T(s, t)\,, \qquad A(s;\, \cos\theta) = A(s, t)\,,$$

the dispersion relations (6), (7) take the form

$$(\varepsilon \to +0,\ t(|\zeta|) = (s - 4m^2)(|\zeta| - 1|)/2)\,.$$

(9) $$T(s;\, \cos\theta) = C(s) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}\zeta\, \frac{\varepsilon(\zeta) A\big(t(|\zeta|),\, s + i\varepsilon\big)}{\zeta - \cos\theta}\,, \qquad s \geqslant 4m^2\,.$$

(10) $$A\,(s,\, \cos\theta) = \varrho\,(s) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}\zeta\, \frac{\varepsilon(\zeta)\varrho\big(s,\, t(|\zeta|)\big)}{\zeta - \cos\theta}\,.$$

T and A are even functions of $\cos\theta$

$$T(s;\, \cos\theta) = T(s;\, -\cos\theta)\,, \qquad A(s;\, \cos\theta) = A(s;\, -\cos\theta)\,.$$

The partial wave expansion

$$T(s;\, \cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, T_l(s) P_l(\cos\theta)\,,$$

$$T_l(s) = \frac{1}{2}\int_{-1}^{+1} \mathrm{d}\cos\theta\; T(s;\, \cos\theta) P_l(\cos\theta)\,,$$

converges in the ellipse of the $\cos\theta$ plane with foci ± 1 and semi major axis

$$a_0(s) = 1 + \frac{8m^2}{s - 4m^2},$$

if $s \geqslant 4m^2$.

2. – The unitarity condition.

In the elastic scattering region the unitarity of the S-matrix implies

$$\text{(11)} \qquad \operatorname{Im} T(k_1 k_2 | k_1' k_2') = \frac{1}{2}\int \mathrm{d}l_1\, \mathrm{d}l_2\, \theta(l_1)\, \mathrm{d}(l_1^2 + m^2)\theta(l_2)\, \mathrm{d}(l_2^2 + m^2)\cdot T(k_1 k_2 | l_1 l_2) T^*(l_1 l_2 | k_1' k_2'),$$

if $4m^2 \leqslant s < 16m^2$, $s = -(k_1+k_2)^2$.

For $s > 16m^2$ additional contributions to the right hand side come from intermediate states with more than two particles. For all $s \geqslant 4m^2$ one can define, however, the two particle projection B of the absorptive part by the right hand side of (11)

$$\text{(12)} \qquad B(k_1 k_2 | k_1' k_2') = \frac{1}{2}\int \mathrm{d}l_1\, \mathrm{d}l_2\, \theta(l_1)\, \mathrm{d}(l_1^2 + m^2)\theta(l_2)\, \mathrm{d}(l_2^2 + m^2)\cdot T(k_1 k_2 | l_1 l_2) T^*(l_1 l_2 | k_1' k_2').$$

In the elastic region B coincides with A

$$\text{(13)} \qquad B(k_1 k_2 | k_1' k_2') = A(k_1 k_2 | k_1' k_2') \qquad \text{for } 4m^2 \leqslant s < 16m^2 .$$

Introducing s and $\cos\theta$ as independent variables

$$B(k_1 k_2 | k_1' k_2') = \delta(k_1 + k_2 - k_1' - k_2') B(s; \cos\theta) .$$

$B(s; \cos\theta)$ satisfies a dispersion relation in $\cos\theta$

$$\text{(14)} \qquad B(s; \cos\theta) = \beta(s) + \frac{1}{\pi}\int \mathrm{d}\zeta\, \frac{\varepsilon(\zeta)\beta(s; |\zeta|)}{\zeta - \cos\theta}, \qquad s \geqslant 4m^2 ,$$

with the weight functions

$$\text{(15)} \qquad \beta(s) = \frac{\pi}{4}\frac{\sqrt{s - 4m^2}}{\sqrt{s}}\{C(s) T_0^*(s) + C^*(s) T_0(s) - |C(s)|^2\},$$

$$\text{(16)} \qquad \beta(s; \zeta) = -\frac{1}{2}\frac{\sqrt{s - 4m^2}}{\sqrt{s}}\int_0^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta\, A(t(|\xi|), s + i\varepsilon)\, A^*(t(|\eta|), s + i\varepsilon)\cdot \frac{\theta(\zeta - \zeta_0)}{\sqrt{k(\xi, \eta, \zeta)}}, \qquad \zeta > 0, \qquad \varepsilon \to +0,$$

$$k(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - 2\xi\eta\zeta - 1 \qquad t(|\xi|) = (s - 4m^2)\frac{|\xi| - 1}{2} .$$

$$\zeta_0 = \xi\eta + \sqrt{\xi^2 - 1}\sqrt{\eta^2 - 1} .$$

Here $T_0(s)$ is the s-wave amplitude, $C(s)$ the constant term (8) in the corresponding dispersion relation (6) of T.

Formulae of the type (14)–(16) have been obtained by MANDELSTAM (M1-3) and play a central part in his work. The derivation will be sketched briefly. We introduce

$$T'(k_1k_2|l_1l_2) = T(k_1k_2|l_1l_2) - C(k_1k_2|l_1l_2) ,$$

with

$$C(k_1k_2|l_1l_2) = \delta(k_1 + k_2 - l_1 - l_2)\, C(-(k_1 + k_2)^2) ,$$

and replace the integrand TT^* on the right hand side of (17) by

$$TT^* = CT^* + C^*T - CC^* + T'T^{*\prime} .$$

The integrals over the first three terms do not depend on $\cos\theta$ and are therefore determined by their s-wave projections. For the first term we obtain for instance

$$\frac{1}{2}\int \mathrm{d}l_1\, \mathrm{d}l_2 \theta(l_1)\delta(l_1^2 + m^2)\theta(l_2)\delta(l_2^2 + m^2)\, C(k_1k_2|l_1\, l_2)\, T^*(l_1\, l_2|k_1'\, k_2') =$$
$$= \frac{\pi}{4}\frac{\sqrt{s - 4m^2}}{\sqrt{s}}\, C(s)\, T_0^*(s)\, \delta(k_1 + k_2 - k_1 - k_2') .$$

Inserting

$$T'(s; \cos\theta) = \frac{1}{\pi}\int \mathrm{d}\zeta\, \frac{\varepsilon(\zeta)A(t|\zeta|, s + i\varepsilon)}{\cos\theta - \zeta} .$$

into the remaining integral over $T'T'^*$ and carrying out the integrations over the angles (M1) we arrive at (14) with the weight functions (15), (16).

We finally discuss some simple properties of $B(s; \cos\theta)$ using the convention $B(s; \cos\theta) = 0$ for all $s < 4m^2$. The weight functions $\beta(s)$ and $\beta(s; z)$ are real and

$$\operatorname{Im} B(s; z) = \varepsilon(z)\beta(s; |z|) \qquad \text{for } z \text{ real,}$$

$\beta(s; z)$ vanishes

$$\beta(s) = \beta(s; z) = 0 \qquad \text{for } s < 4m^2$$

and

(17) $$\beta(s; z) = 0 \qquad \text{if } |z| < 2a_0^2(s) - 1, \quad s \geqslant 4m^2,$$

$$a_0(s) = 1 + \frac{8m^2}{s - 4m^2} .$$

The partial wave expansion

$$B(s;\,\cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, B_l(s)\, P_l(\cos\theta)$$

converges inside the ellipse with foci ± 1 and semi major axis $2a_0(s)^2 - 1$, the coefficients being given by

$$B_l(s) = \frac{\pi}{4}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,|T_l(s)|^2 \leqslant A_l(s)\,. \tag{18}$$

As function of s and t

$$\beta(s,t) = \theta(z)\beta(s;\,|z|)\,, \qquad |z| = 1 + \frac{2t}{s-4m^2}\,,$$

is a real function and vanishes (M3)

$$\left\{\begin{array}{llll} \beta(s,t)=0 & \text{if} & s<4m^2\,, & \\ \text{and} & & & \\ \beta(s,t)=0 & \text{if} & t<16m^2+\dfrac{64m^4}{s-4m^2} & (s\geqslant 4m^2)\,. \end{array}\right. \tag{19}$$

In terms of s and t eq. (14) takes the form

$$B(s,t) = \beta(s) + \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}t'\,\frac{\beta(s,t')}{t'-t} + \frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}u'\,\frac{\beta(s,u')}{u'-u}\,. \tag{20}$$

Comparing (20) with the corresponding dispersion relation (7) of the absorptive part we obtain from (16)

$$\beta(s) = \varrho(s)\,, \qquad \beta(s,t) = \varrho(s,t) \qquad \text{if } 4m^2 \leqslant s < 16m^2. \tag{21}$$

This is Mandelstam's result that the spectral function $\varrho(s,t)$ can be written as a bilinear expression (16) of the absorptive part if one of the variable lies in the elastic region (M1).

3. – Two particle structure of the scattering amplitude.

In Section **2** we have investigated the properties of the two particle projection $B(s,t)$ of the absorptive part. $B(s,t)$ coincides with the full absorptive part in the elastic region of s and satisfies a dispersion relation in t with the

weight functions $\beta(s)$, $\beta(s,t)$ given by (15), (16). We will now use $\beta(s)$ and $\beta(s,t)$ for constructing an amplitude $\Phi_{el}(s,t)$ which satisfies the Mandelstam representation and has exactly the same discontinuities as $\Phi(s,t)$ across the elastic portions of the s, t and u-cut. We define $\Phi_{el}(s,t)$ by the Mandelstam representation

$$\begin{aligned}(22)\qquad \Phi_{el}(s,t) = \lambda + \frac{1}{\pi}\int ds'\,\frac{\varrho_{el}(s')}{s'-s} + \frac{1}{\pi}\int dt'\,\frac{\varrho_{el}(t')}{t'-t} + \frac{1}{\pi}\int du'\,\frac{\varrho_{el}(u')}{u'-u} + \\ + \frac{1}{\pi^2}\int ds'\,dt'\,\frac{\varrho_{el}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int dt'\,du'\,\frac{\varrho_{el}(t',u')}{(t'-t)(u'-u)} + \\ + \frac{1}{\pi^2}\int du'\,ds'\,\frac{\varrho_{el}(u',s')}{(u'-u)(s'-s)}\,,\end{aligned}$$

$$u = 4m^2 - s - t\,,$$

with the spectral functions

$$(23)\qquad \begin{cases}\varrho_{el}(s) = \beta(s)\,,\\ \varrho_{el}(s,t) = \beta(s,t) + \beta(t,s)\,,\end{cases}$$

Φ_{el} will be called the elastic part of the scattering amplitude. It satisfies the following conditions:

i) Φ_{el} is symmetric in the variables s, t, and u;

ii) across each cut the elastic part Φ_{el} and the scattering amplitude have the same discontinuity in the elastic region

$$(24)\qquad \begin{cases}A_{el}(s,t) = A(s,t)\,, \qquad 4m^2 \leqslant s < 16m^2\,,\\ \text{with}\\ A_{el}(s,t) = \dfrac{1}{2i}\lim\limits_{\varepsilon\to+0}\left(\Phi_{el}(s+i\varepsilon,t) - \Phi_{el}(s-i\varepsilon,t)\right)\,;\end{cases}$$

iii) the spectral functions of the representation of Φ_{el} and Φ are identical if at least one variable lies in the elastic region

$$(25)\qquad \begin{cases}\varrho_{el}(s) = \varrho(s) \quad \text{if} \quad 4m^2 \leqslant s < 16m^2\,,\\ \varrho_{el}(s,t) = \varrho(s,t) \quad \text{if} \quad 4m^2 \leqslant s < 16m^2 \quad \text{or} \quad 4m^2 \leqslant t < 16m^2\,;\end{cases}$$

i) follows immediately from

$$\varrho_{el}(s,t) = \varrho_{el}(t,s)\,,$$

and the symmetric form of the defining representation (22);

ii) follows from

$$(26)\qquad A_{\text{el}}(s,t)=\varrho_{\text{el}}(s)+\frac{1}{\pi}\int \mathrm{d}t'\,\frac{\varrho_{\text{el}}(s,t')}{t'-t}+\frac{1}{\pi}\int \mathrm{d}u'\,\frac{\varrho_{\text{el}}(s,u')}{u'-u}=$$

$$=B(s,t)+\frac{1}{\pi}\int\limits_{4m^2}^{\infty}\mathrm{d}t'\,\frac{\beta(t',s)}{t'-t}+\frac{1}{\pi}\int\limits_{4m^2}^{\infty}\mathrm{d}u'\,\frac{\beta(u',s)}{u'-u}\quad(\text{eq. }(20))\,.$$

As the two integrals on the right hand side vanish for $s<16m^2$ we have

$$A_{\text{el}}(s,t)=B(s,t)=A(s,t)\qquad\text{if}\qquad 4m^2\leqslant s<16m^2\,.$$

For the proof of iii) let us first consider $\varrho_{\text{el}}(s,t)$ for s in the elastic region. We then have

$$\varrho_{\text{el}}(s,t)=\beta(s,t)=\varrho(s,t)\qquad\text{if}\qquad 4m^2\leqslant s<16m^2\,,$$

because (23), (19) and (21). If t lies in the elastic region we have

$$\varrho_{\text{el}}(s,t)=\beta(t,s)=\varrho(s,t)\qquad\text{if}\qquad 4m^2\leqslant s<16m^2\,.$$

Representation (22) can also be written in the form

$$(27)\qquad \Phi_{\text{el}}(s,t)=\lambda+L(s,t)+L(t,u)+L(u,s)$$

with

$$(28)\qquad L(s,t)=\frac{1}{\pi}\int\limits_{4m^2}^{\infty}\mathrm{d}s'\,\frac{\beta(s')}{s'-s}+\frac{1}{\pi^2}\int\limits_{4m^2}^{\infty}\mathrm{d}s'\int\limits_{c(s')}^{\infty}\mathrm{d}t'\,\frac{\beta(s',t')}{(s'-s)(t'-t)}+$$

$$+\frac{1}{\pi^2}\int\limits_{4m^2}^{\infty}\mathrm{d}s'\int\limits_{c(s')}^{\infty}\mathrm{d}u'\,\frac{\beta(s',u')}{(s'-s)(u'-u)}\,,$$

$$u=4m^2-s-t\,,\qquad c(s')=16m^2+\frac{64m^4}{s'-4m^2}\,,$$

L is symmetric in t and u

$$L(s,t)=L(s,4m^2-s-t)$$

and an analytic function of s and t except for the cuts

$$s\geqslant 4m^2\,,\qquad t\geqslant 16m^2\,,\qquad u\geqslant 16m^2\,.$$

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Across the cuts L has the discontinuities

$$\lim_{\varepsilon\to+0}\big(L(s+i\varepsilon,t)-L(s-i\varepsilon,t)\big)=2iB(s,t) \qquad \text{for} \qquad s\geqslant 4m^2\,,$$

$$\lim_{\varepsilon\to+0}\big(L(s,t+i\varepsilon)-L(s,t-i\varepsilon)\big)=\frac{2i}{\pi}\int_{4m^2}^{\infty}\mathrm{d}s'\,\frac{\beta(s',t)}{s'-s} \qquad \text{for} \qquad t\geqslant 16m^2\,.$$

With the results obtained so far it is possible to separate the two-particle contributions from the scattering amplitude in all three reaction channels. Combining (1) and (22) we obtain

$$(29)\qquad \Phi(s,t)=\Phi_{\mathrm{el}}(s,t)+\Phi_{\mathrm{inel}}(s,t) \qquad \begin{aligned}\varrho(s)&=\varrho_{\mathrm{el}}+\varrho_{\mathrm{inel}}(s)\,,\\ \varrho(s,t)&=\varrho_{\mathrm{el}}(s,t)+\varrho_{\mathrm{inel}}(s,t)\,,\end{aligned}$$

where Φ_{el} as well as Φ_{inel} satisfy a spectral representation with full symmetry in s, t, and u. The spectral function ϱ_{inel} of the inelastic part Φ_{inel} are defined by

$$(30)\qquad \varrho_{\mathrm{inel}}(s)=\varrho(s)-\varrho_{\mathrm{el}}(s)\,,\qquad \varrho_{\mathrm{inel}}(s,t)=\varrho(s,t)-\varrho_{\mathrm{el}}(s,t)\,,$$

$\varrho_{\mathrm{inel}}(s,t)$ is symmetric

$$(31)\qquad \varrho_{\mathrm{inel}}(s,t)=\varrho_{\mathrm{inel}}(t,s)\,.$$

As consequence of the properties ii), iii) ϱ_{inel} and the absorptive part A_{inel} of Φ_{inel} vanish identically in the elastic region

$$(32)\qquad \left|\begin{array}{lll}\varrho_{\mathrm{inel}}(s)\equiv 0 & \text{if} & s\leqslant 16m^2\,,\\ \varrho_{\mathrm{inel}}(s,t)\equiv 0 & \text{if} & s\leqslant 16m^2 \ \text{or}\ t\leqslant 16m^2\,,\\ A_{\mathrm{inel}}(s,t)\equiv 0 & \text{if} & 4m^2\leqslant s<16m^2\,,\end{array}\right.$$

where

$$2iA_{\mathrm{inel}}(s,t)=\lim_{\varepsilon\to+0}\big(\Phi_{\mathrm{inel}}(s+i\varepsilon,t)-\Phi_{\mathrm{inel}}(s-i\varepsilon,t)\big) \qquad s\geqslant 4m^2\,.$$

The inelastic part Φ_{inel} is therefore analytic in s and t except for the three cuts in s, t and u starting at the first inelastic threshold $16m^2$

$$s\geqslant 16m^2\,,\qquad t\geqslant 16m^2\,,\qquad u\geqslant 16m^2\,.$$

Above $s=16m^2$ the unitarity condition implies for the inelastic part

$$(33)\qquad A_{\mathrm{inel}}(s,t)=Q(s,t)-\frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}t'\,\frac{\beta(t',s)}{t'-t}-\frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}u'\,\frac{\beta(u',s')}{u'-u}\,,$$

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with

$$Q(s, t) = \frac{1}{2}\sum_{n=4}^{\infty}\int Dl_1 \dots Dl_n T(k_1 k_2 | l_1 \dots l_n) T^*(l_1 \dots l_n | k_1' k_2') ,$$

$$i T(k_1 k_2 | l_1 \dots l_n) = (\Phi^{\text{out}}_{k_1 k_2}, \Phi^{\text{in}}_{l_1 \dots l_n}) , \qquad n \geqslant 4 , \qquad Dl = \mathrm{d}l\,\theta(l)\delta(l^2 + m^2) ,$$

$$s = -(k_1 + k_2) , \qquad t = -(k_1 - k_1')^2 , \qquad u = -(k_1 - k_2')^2 .$$

The summation on the right hand side starts with the four particle intermediate states. The two additional terms cancel the discontinuity of Q across the elastic portion of the t- and u-cut

$$\lim_{\varepsilon \to +0} \big(Q(s, t + i\varepsilon) - Q(s, t - i\varepsilon)\big) = 2i\beta(t, s) \qquad \text{for} \qquad 4m^2 \leqslant t < 16m^2 .$$

4. – Integral equations.

In this section we study the scattering amplitude for fixed inelastic spectral functions ϱ_{inel}. Let $\varrho_{\text{inel}}(s)$ and $\varrho_{\text{inel}}(s, t)$ be given arbitrarily, vanishing outside $s \geqslant 16m^2$, $t \geqslant 16m^2$. For this case we will derive integral equations as necessary conditions for the weight functions β of the two particle projection B of the absorptive part (eq. (14)). In Section **2** we obtained for β the equations

$$(34a)\quad \beta(s) = \frac{\pi}{4}\,\theta(s - 4m^2)\,\frac{\sqrt{s - 4m^2}}{\sqrt{s}}\left\{ |C(s)|^2 + \frac{C^*(s)}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\zeta\, \varepsilon(\zeta) A\big(t(|\zeta|), s + i\varepsilon\big) \lg \frac{\zeta + 1}{\zeta - 1} + \right.$$

$$\left. + \frac{C(s)}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\zeta\, \varepsilon(\zeta) A^*(t(|\zeta|), s + i\varepsilon) \lg \frac{\zeta + 1}{\zeta - 1}\right\} , \qquad \varepsilon \to +0$$

$$(34b)\qquad \beta(s, \zeta) = -\frac{1}{2}\,\theta(s - 4m^2)\,\frac{\sqrt{s - 4m^2}}{\sqrt{s}}\int_0^{\infty} \mathrm{d}\xi \int_0^{\infty} \mathrm{d}\eta\, A\,(t(|\xi|), s + i\varepsilon) \cdot$$

$$\cdot A^*(t(|\eta|), s + i\varepsilon)\,\frac{\theta(\zeta - \zeta_0)}{\sqrt{k(\xi, \eta, \zeta)}} \qquad \varepsilon \to +0 ,$$

$$k(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - 2\xi\eta\zeta - 1 , \qquad \beta(s; \zeta) = \beta(s, t(|\zeta|)) , \qquad \zeta > 0 ,$$

$$\zeta_0 = \xi\eta + \sqrt{\xi^2 - 1}\,\sqrt{\eta^2 - 1} ,$$

holding for any s, ζ real. (34a) follows from (15) by inserting

$$T_0(s) = C(s) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\zeta\, \varepsilon(\zeta) A(t(|\zeta|), s + i\varepsilon) \lg \frac{\zeta + 1}{\zeta - 1} .$$

On the right hand side of (34) there occur the functions $C(s)$ and $A(t(|\zeta|), s+i\varepsilon)$ which can themselves be explicitly expressed in terms of λ, β, and ϱ_{inel} (eq. (8), (7), (29) and (23))

$$C(s) = \lambda + \int_{16m^2}^{\infty} ds' \frac{\varrho_{\text{inel}}(s')}{s'-s-i\varepsilon} + \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{\beta(s')}{s'-s-i\varepsilon}, \qquad \varepsilon \to +0, \tag{34'}$$

$$A(t, s+i\varepsilon) = 0 \qquad \text{for } t < 4m^2,$$

$$A(t, s+i\varepsilon) = \varrho_{\text{inel}}(t) + \beta(t) + \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{\varrho_{\text{inel}}(t, s') + \beta(t, s') + \beta(s', t)}{s'-s-i\varepsilon} + \frac{1}{\pi}\int_{4m^2}^{\infty} du' \frac{\varrho_{\text{inel}}(t, u') + \beta(t, u') + \beta(u', t)}{u'-u}, \tag{34''}$$

with

$$\varepsilon \to +0, \qquad u = 4m^2 - s - t, \qquad \text{for } t \geqslant 4m^2.$$

Substituting (34'), (34'') for C and A on the right hand sides of (34a), (34b) we get two coupled non-linear integrals equations for $\beta(s)$ and $\beta(s, t)$ involving λ, $\varrho_{\text{inel}}(s)$ and $\varrho_{\text{inel}}(s, t)$ as given parameters [1,2,4].

Let $\beta(s)$, $\beta(s, t)$ be solutions of these integral eq. (34) for ϱ_{inel} symmetric and vanishing in the elastic regions. Then the amplitude Φ constructed by the spectral representation with

$$\begin{cases} \varrho(s) = \beta(s) + \varrho_{\text{inel}}(s), \\ \varrho(s, t) = \beta(s, t) + \beta(t, s) + \varrho_{\text{inel}}(s, t), \end{cases} \tag{35}$$

is crossing symmetric and satisfies the unitarity condition in the elastic regions. Hence the eq. (34) form necessary and sufficient conditions and we have the following theorems.

I. If the amplitude Φ is crossing symmetric and satisfies the unitarity condition in the elastic region the functions β (defined by (12), (14)) satisfy the integral eq. (34) with ϱ_{inel} real symmetric and vanishing in the elastic regions.

II. If the functions β are solutions of the integral eq. (34) with ϱ_{inel} real symmetric and vanishing in the elastic regions the amplitude Φ defined by (1), (35) is crossing symmetric and satisfies the unitarity condition in the elastic region.

If the inelastic spectral functions ϱ_{inel} are set equal to zero

$$\varrho_{\text{inel}}(s) = \varrho_{\text{inel}}(s, t) = 0 ,$$

(34) represent the integral equations of the two particle approximation [1-4]. Crossing symmetry and spectral representation are guaranteed in this approximation. Instead of the unitarity condition one obtains from (26)

$$\text{(36)} \qquad \operatorname{Im} T(s, t) = B(s, t) + \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}t' \, \frac{\beta(t', s)}{t'-t} + \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}u' \, \frac{\beta(u', s)}{u'-u} \qquad \begin{array}{l} s \geqslant 4m^2 \\ 0 \leqslant -t \leqslant s - 4m^2 . \end{array}$$

Here $B(s, t)$ is the expected two particle contribution (12), the remaining two integrals vanish for $s < 16m^2$ and replace the proper four and more particle contributions of the exact unitarity condition.

5. – Extension to other models.

The formalism developed in this paper is not restricted to the particular model studied in the preceding sections. Neither crossing symmetry nor equal masses are essential as long as the Mandelstam representation holds and the two particle regions considered lie within the physical domain of the incoming and outgoing particles [10]. In this section we shall discuss the following models:

a) the scattering of (not identical) equal mass particles as a model without crossing symmetry;

b) the scattering of equal mass particles with non vanishing vertex function;

c) meson-nucleon scattering (spins disregarded), for this case we treat the two particle structure for the two meson-nucleon channels only.

a) *No crossing symmetry.* We consider the scattering of equal mass particles but now without assuming any symmetry properties. (The vertex function is again assumed to be zero.) Then the Mandelstam representation contains six different spectral functions ϱ_1, ϱ_2, ϱ_3, ϱ_{12}, ϱ_{13}, ϱ_{23} with the indices 1, 2, 3 referring to the variables s, u, and t. The discontinuities across the s-, u- and t-cut will be as usual denoted by A_1, A_2 and A_3. Instead of (12)

[10] This is different, for instance, in the case of nucleon-antinucleon annihilation.

we define three two-particle projections B_1, B_2 and B_3 of the absorptive parts. It is

$$B_1(s,t)=A_1(s,t) \quad \text{for} \quad 4m^2\leqslant s<16m^2\,,$$

$$B_2(u,s)=A_2(u,s) \quad \text{for} \quad 4m^2\leqslant u<16m^2\,,$$

$$B_3(t,s)=A_3(t,s) \quad \text{for} \quad 4m^2\leqslant t<16m^2\,.$$

Instead of (14), (20) we obtain for B_i

$$B_1(s,t)=\beta_1(s)+\frac{1}{\pi}\int\mathrm{d}u'\,\frac{\beta_{12}(s,u')}{u'-u}+\frac{1}{\pi}\int\mathrm{d}t'\,\frac{\beta_{13}(s,t')}{t'-t}\,,$$

$$B_2(u,s)=\beta_2(u)+\frac{1}{\pi}\int\mathrm{d}t'\,\frac{\beta_{23}(u,t')}{t'-t}+\frac{1}{\pi}\int\mathrm{d}s'\,\frac{\beta_{21}(u,s')}{s'-s}\,,$$

$$B_3(t,s)=\beta_3(t)+\frac{1}{\pi}\int\mathrm{d}s'\,\frac{\beta_{31}(t,s')}{s'-s}+\frac{1}{\pi}\int\mathrm{d}u'\,\frac{\beta_{32}(t,u')}{u'-u}\,,$$

where the β_{ij} are bilinear expressions in the absorptive parts ([11]). The β_{ij} vanish in the following regions

$$\beta_{12}(s,u)=0 \quad \text{for} \quad u<16m^2+\frac{64m^4}{s-4m^2} \qquad (s\geqslant 4m^2)\,,$$

$$\beta_{21}(u,s)=0 \quad \text{for} \quad s<16m^2+\frac{64m^4}{s-4m^2} \qquad (u\geqslant 4m^2)\,,\ \text{etc.}\,.$$

We further have

$$\varrho_{12}(s,u)=\beta_{12}(s,u) \quad \text{for} \quad 4m^2\leqslant s<16m^2$$

$$\varrho_{12}(s,u)=\beta_{21}(u,s) \quad \text{for} \quad 4m^2\leqslant u<16m^2 \quad \text{etc.}$$

and similar relations for the other functions.

The elastic part Φ_{el} can now be defined by the Mandelstam representation with the spectral function

$$\varrho_i^{\mathrm{el}}=\beta_i\,,\qquad \varrho_{ij}^{\mathrm{el}}(u_iu_j)=\beta_{ij}(u_iu_j)+\beta_{ji}(u_ju_i)\,.$$

Then

$$\varrho_{12}^{\mathrm{el}}(s,u)=\beta_{12}(s,u)=\varrho_{12}(s,u) \quad \text{if} \quad 4m^2\leqslant s<16m^2\,,$$

$$\varrho_{12}^{\mathrm{el}}(s,u)=\beta_{21}(u,s)=\varrho_{12}(s,u) \quad \text{if} \quad 4m^2\leqslant u<16m^2\,.$$

Combined with the corresponding relations for the other $\varrho_{ij}^{\mathrm{el}}$ this is sufficient to verify all results of Section **3** except the symmetry properties.

([11]) See eq. (49) of Section **5**c with $K(s)=\sqrt{(s-4m^2)}/2$.

b) Non vanishing vertex function. Returning to the scattering amplitude of identical particles we now indicate the modifications necessary in the general case of non vanishing three particle vertex. Instead of (4) we have

$$\varrho\,(s) = \pi g^2\,\delta(s-m^2) + \varrho'(s)\,,$$
$$\varrho'(s) = 0 \qquad\qquad \text{for}\ \ s < 4m^2.$$

The dispersion relation (9) of T then takes the form

$$T(s;\cos\theta) = C(s) + \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \mathrm{d}\zeta\, \frac{\varepsilon(\zeta)\, A\big(t(|\zeta|),\, s+i\varepsilon\big)}{\zeta-\cos\theta}\,, \quad s \geqslant 4m^2, \quad \varepsilon \to +0\,, \tag{37}$$

with

$$A\,(t,s) = \pi g^2\,\delta(t-m^2) + A'(t,s)\,,$$
$$A'(t,s) = \theta(t-4m^2)\,A(t,s)\,.$$

The definition (12) of B remains unchanged. Inserting (37) into the right hand side of (12) we obtain (14) with the weight functions

$$\left\{\begin{aligned}
&\beta(s) = \frac{\pi}{4}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\{C(s)\,T_0^*(s) + C^*(s)\,T_0(s) - |C(s)|^2\} \qquad \text{for}\ s \geqslant 4m^2\,,\\
&\beta(s,t) = \varrho^{(4)}(s,t) + \beta'(s,t)\,,\\
&\beta'(s,t) = -\frac{1}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\int\limits_0^\infty \mathrm{d}\xi \int\limits_0^\infty \mathrm{d}\eta\, C(s;\xi\eta)\,\frac{\theta(\zeta-\zeta_0)}{\sqrt{k(\xi,\eta,\zeta)}}\,, \qquad t=(s-4m^2)\,\frac{|\zeta|-1}{2}\,,\\
&\qquad\qquad \text{for}\ \ s \geqslant 4m^2\,,\quad t \geqslant 9m^2\,.\\
&C(s;\xi\eta) = A'(t_1,\, s+i\varepsilon)\,A'^*(t_2,\, s+i\varepsilon) + g^2\,\delta(t_1-m^2)\,A'^*(t_2,\, s+i\varepsilon)\, +\\
&\qquad\qquad + g^2\delta(t_2-m^2)\,A'(t_1,\, s+i\varepsilon)\,,\\
&\left(t_1 = (s-4m^2)\,\frac{|\xi|-1}{2}\,, \qquad t_2 = (s-4m^2)\,\frac{|\eta|-1}{2}\right) \qquad \varepsilon \to +0\,,
\end{aligned}\right. \tag{38}$$

$\varrho^{(4)}$ is just the fourth order contribution to the spectral function for the A^3-coupling (M2)

$$\varrho^{(4)}(s,t) = -\frac{\pi^2}{2}\frac{\sqrt{s-4m^2}}{\sqrt{s}}\int \mathrm{d}\xi\,\mathrm{d}\eta\, g^4\,\delta(t_1-m^2)\,\delta(t_2-m^2)\,\frac{\theta(\zeta-\zeta_0)}{\sqrt{k(\xi,\eta,\zeta)}} =$$
$$= -2g^4\pi^2\,\frac{\theta\big(K(s,t)\big)}{\sqrt{K(s,t)}}\,, \quad \text{for}\ \ s \geqslant 4m^2\,,\quad t \geqslant 4m^2\,,$$

$$K(s,t) = 4st\big(st - 4m^2(s+t) + 12m^4\big)\,.$$

In the elastic region of the s-channel holds

$$\varrho(s, t) = \beta(s, t) = \varrho^{(4)}(s, t) + \beta'(s, t) \quad \text{if} \quad 4m^2 \leqslant s < 9m^2 ,$$

where

$$\beta'(s, t) = 0 \quad \text{if} \quad s < 4m^2 \quad \text{or} \quad t < 9m^2$$

with the boundary curve determined by (M3)

$$(t - 9m^2)(t - m^2)(s - 4m^2) - 16m^4 t = 0$$

(asymptotes $s = 4m^2$, $t = 9m^2$). The region where $\varrho^{(4)}$ is non-vanishing, however, extends into the strip $4m^2 < t < 9m^2$. This is the reason why one does not succeed simply by symmetrizing $\beta(s, t)$. Instead we define the elastic part Φ_{el} by the spectral functions

$$\text{(39)} \qquad \begin{cases} \varrho_{el}(s) = \pi g^2 \, \delta(s - m^2) + \theta(s - 4m^2)\beta(s) , \\ \varrho_{el}(s, t) = \varrho^{(4)}(s, t) + \beta'(s, t) + \beta'(t, s) . \end{cases}$$

As

$$\varrho_{el}(s, t) = \varrho^{(4)}(s, t) + \beta'(s, t) = \varrho(s, t)$$

for

$$4m^2 \leqslant s < 9m^2 \; (t \geqslant 4m^2)$$

the elastic part Φ_{el} satisfies the three conditions given in Section **3**. We further have

$$\Phi(s, t) = \Phi_{el}(s, t) + \Phi_{inel}(s, t) ,$$

where the spectral functions ϱ_{inel} of the inelastic part Φ_{inel} vanish in the elastic regions

$$\varrho_{inel}(s) = 0 \quad \text{if} \quad s < 9m^2 ,$$
$$\varrho_{inel}(s, t) = 0 \quad \text{if} \quad s < 9m^2 \quad \text{or} \quad t < 9m^2 .$$

As a consequence the total spectral function can be decomposed in the form

$$\text{(40)} \qquad \varrho(s, t) = \varrho^{(4)}(s, t) + \beta'(s, t) + \beta'(t, s) + \varrho_{inel}(s, t)$$

with $\varrho^{(4)}$, β' and ϱ_{inel} vanishing in the regions indicated above.

c) Meson-nucleon scattering. We finally discuss the scattering of a charged spinless « nucleon » of mass M with initial momentum p, final momentum p' and a neutral, spinless « meson » of mass $m < M/2$ with initial momentum k, final momentum k'. Stable bound states and any coupling to other particles are excluded, the three meson vertex function is assumed to be zero.

The spectral functions of the Mandelstam representation

$$\Phi(s,t) = \frac{1}{\pi}\int \mathrm{d}s' \frac{\varrho_1(s')}{s'-s} + \frac{1}{\pi}\int \mathrm{d}u' \frac{\varrho_2(u')}{u'-u} + \frac{1}{\pi}\int \mathrm{d}t' \frac{\varrho_3(t')}{t'-t} + \frac{1}{\pi^2}\int \mathrm{d}s'\,\mathrm{d}u' \frac{\varrho_{12}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\int \mathrm{d}u'\,\mathrm{d}t' \frac{\varrho_{23}(u',t')}{(u'-u)(t'-t)} + \frac{1}{\pi^2}\int \mathrm{d}t'\,\mathrm{d}s' \frac{\varrho_{13}(t',s')}{(t'-t)(s'-s)}, \tag{41}$$

$$u + s + t = 2M^2 + 2m^2,$$

have the following structure

$$\left\{\begin{aligned} \varrho_1(s) &= \varrho_2(s) = \pi g^2\,\delta(s-M^2) + \varrho_1'(s), \\ \varrho_1'(s) &= 0 \quad \text{for} \quad s < (M+m)^2, \\ \varrho_3(t) &= 0 \quad \text{for} \quad t < 4m^2, \\ \varrho_{12}(s,u) &= 0 \quad \text{for} \quad s < (M+m)^2 \quad \text{or} \quad u < (M+m)^2, \\ \varrho_{23}(s,t) &= \varrho_{13}(s,t) = 0 \quad \text{for} \quad s < (M+m)^2 \ \text{or} \ t < 4m^2. \end{aligned}\right. \tag{42}$$

Φ is symmetric in s and u (crossing symmetry)

$$\left\{\begin{aligned} \Phi(s,t) &= \Phi(u,t), \qquad u = 2M^2 + 2m^2 - s - t, \\ \varrho_{12}(s,u) &= \varrho_{12}(u,s), \\ \varrho_{13}(s,t) &= \varrho_{23}(s,t). \end{aligned}\right. \tag{43}$$

For the absorptive part we use the notation

$$\left\{\begin{aligned} A_1(s,t) &= \pi g^2\,\delta(s-M^2) + \frac{1}{2i}\,\theta(s-(M+m)^2)\cdot \\ &\qquad \cdot \lim_{\varepsilon\to+0} (\Phi(s+i\varepsilon,t) - \Phi(s-i\varepsilon,t)) \qquad \text{for } s \text{ real}, \\ A_2(u,s) &= A_1(u,t), \qquad t = 2M^2 + 2m^2 - s - u, \\ A_3(t,s) &= \theta(t-4m^2) \lim_{\varepsilon\to+0} (\Phi(s,t+i\varepsilon) - \Phi(s,t-i\varepsilon)) \qquad \text{for } t \text{ real}. \end{aligned}\right. \tag{44}$$

In the elastic region of meson-nucleon scattering the unitarity of the S-matrix implies

$$\text{(45)}\quad \operatorname{Im} T(pk\,|\,p'k') = \frac{1}{2}\int \mathrm{d}l_1 \mathrm{d}l_2 \theta(l_1)\theta(l_2)\delta(l_1^2+M^2)\delta(l_2^2+m^2)T(pk\,|\,l_1 l_2)T^*(l_1 l_2\,|\,p'k'),$$

$$(M+m)^2 \leqslant s < (M+2m)^2 .$$

In order to study the two particle structure of the scattering amplitude for the meson-nucleon reactions we define

$$\text{(46)}\qquad \delta(p+k-p'-k')B_1(s,t) = $$
$$= \frac{1}{2}\int \mathrm{d}l_1\, \mathrm{d}l_2 \theta(l_1)\theta(l_2)\,\delta(l_1^2+M^2)\,\delta(l_2^2+m^2)\,T(pk\,|\,l_1 l_2)\,T^*(l_1 l_2\,|\,p'k') .$$

for all $\qquad s = -(p+k)^2 \geqslant (M+m)^2 ,$

and $\qquad B_2(u,s) = B_1(u,t) , \qquad t = 2M^2 + 2m^2 - u - s .$

T denotes the meson-nucleon scattering amplitude

$$T(s,t) = \lim_{\varepsilon\to+0} \Phi(s+i\varepsilon, t) \qquad\qquad (s \geqslant (M+m)^2).$$

In the elastic region B coincides with the absorptive part

$$\text{(47)}\qquad \begin{cases} B_1(s,t) = A_1(s,t) , & (M+m)^2 \leqslant s < (M+2m)^2 , \\ B_2(u,s) = A_2(u,s) , & (M+m)^2 \leqslant u < (M+2m)^2 . \end{cases}$$

For fixed $s \geqslant (M+m)^2$ the B satisfy the dispersion relations

$$\text{(48)}\qquad \begin{cases} B_1(s,t) = \beta_1(s) + \dfrac{1}{\pi}\displaystyle\int \mathrm{d}u' \,\dfrac{\beta_{12}(s,u')}{u'-u} + \dfrac{1}{\pi}\displaystyle\int \mathrm{d}t' \,\dfrac{\beta_{13}(s,t')}{t'-t} , \\ B_2(u,s) = \beta_2(u) + \dfrac{1}{\pi}\displaystyle\int \mathrm{d}t' \,\dfrac{\beta_{23}(u,t')}{t'-t} + \dfrac{1}{\pi}\displaystyle\int \mathrm{d}s' \,\dfrac{\beta_{21}(u,s')}{s'-s} . \end{cases}$$

The weight functions are given by

$$\beta_2 = \beta_1 , \qquad \beta_{21} = \beta_{12} , \qquad \beta_{23} = \beta_{13} ,$$

$$\text{(49a)}\qquad \beta_1(s) = \frac{\pi}{2}\,\theta(s-(M+m)^2)\,\frac{K(s)}{\sqrt{s}}\,\{C(s)T_0^*(s) + C^*(s)T_0(s) - |C(s)|^2\} ,$$

$$(49b)\qquad \beta_{12}(s,u) = -\frac{1}{2}\,\theta(s-(M+m)^2)\theta(-\zeta)\,\frac{K(s)}{\sqrt{s}}\cdot$$

$$\cdot\int \mathrm{d}\xi\,\mathrm{d}\eta\, A_2(u(\xi), s+i\varepsilon)A_3^*(t(\eta), s+i\varepsilon)\,\frac{\theta(|\zeta|-\zeta_0)}{\sqrt{k(\xi,\eta,\zeta)}} - \frac{1}{2}\,\theta(s-(M+m)^2)\theta(-\zeta)\,\frac{K(s)}{\sqrt{s}}\cdot$$

$$\cdot\int \mathrm{d}\xi\,\mathrm{d}\eta\, A_3(t(\xi), s+i\varepsilon)\,A_2^*(u(\eta), s+i\varepsilon)\,\frac{\theta(|\zeta|-\zeta_0)}{\sqrt{k(\xi,\eta,\zeta)}}\,,$$

$$\zeta = -1 - \frac{u}{2K(s)^2} + \frac{(M^2-m^2)^2}{2sK(s)^2}\,,\qquad \varepsilon\to+0\,,$$

$$(49c)\qquad \beta_{13}(s,t) = -\frac{1}{2}\,\theta(s-(M+m)^2)\theta(\zeta)\,\frac{K(s)}{\sqrt{s}}\cdot$$

$$\cdot\int \mathrm{d}\xi\,\mathrm{d}\eta\, A_3(t(\xi), s+i\varepsilon)\,A_3^*(t(\eta), s+i\varepsilon)\,\frac{\theta(|\zeta|-\zeta_0)}{\sqrt{k(\xi,\eta,\zeta)}} - \frac{1}{2}\,\theta(s-(M+m)^2)\theta(\zeta)\,\frac{K(s)}{\sqrt{s}}\cdot$$

$$\cdot\int \mathrm{d}\xi\,\mathrm{d}\eta\, A_2(u(\xi), s+i\varepsilon)\,A_2^*(u(\eta), s+i\varepsilon)\,\frac{\theta(|\zeta|-\zeta_0)}{\sqrt{k(\xi,\eta,\zeta)}}\,,$$

$$\zeta = 1 + \frac{t}{2K(s)^2}\,,\qquad \varepsilon\to+0\,.$$

$$\zeta_0 = |\xi\eta| + |\sqrt{\xi^2-1}\,\sqrt{\eta^2-1}\,,$$

In these formulae $T_0(s)$ is the s-wave amplitude and

$$C(s) = \frac{1}{\pi}\int \mathrm{d}s'\,\frac{\varrho_1(s')}{s'-s-i\varepsilon}\,,\qquad \varepsilon\to+0\,,$$

$$t(\xi) = -2K(s)^2(1-\xi)\,,\qquad u(\xi) = -2K(s)^2(1+\xi) + \frac{(M^2-m^2)^2}{s}\,,$$

$K(s)$ is the center of mass momentum

$$K(s)^2 = \frac{(s-(M+m)^2)(s-(M-m)^2)}{4s}\,.$$

Because (44) the integrations in the first term of (49c) range from $1+(4m^2/2K(s)^2)$ to $+\infty$, in the second term from $-\infty$ to $-1-(4m^2/2K(s)^2)+$ $+((M^2-m^2)^2)/2sK(s)^2$. In the two terms of eq. (49b) the integration parameters have opposite sign. As follows from (44) and (49c) β_{12} vanishes outside $s>(M+m)^2$, $u>(M+2m)^2$,

$$\beta_{12}(s,u) = 0 \qquad \text{if } s<(M+m)^2 \text{ or } u<(M+2m)^2.$$

The region where β_{12} vanishes is bounded by a curve with the asymptotes $s=(M+m)^2$, $u=(M+2m^2)$, (see M3).

Comparing (48) with

$$A_1(s,t)=\varrho_1(s)+\frac{1}{\pi}\int du'\,\frac{\varrho_{12}(s,u')}{u'-u}+\frac{1}{\pi}\int dt'\,\frac{\varrho_{13}(s,t')}{t'-t}\,, \tag{50}$$

we obtain from (47)

$$\begin{aligned}
&\varrho_2(s) = \varrho_1(s)=\beta_1(s) && \text{if}\quad (M+m)^2<s<(M+2m)^2\,,\\
&\varrho_{12}(s,u)=\beta_{12}(s,u) && \text{if}\quad (M+m)^2<s<(M+2m)^2\,,\\
&\varrho_{12}(s,u)=\beta_{12}(u,s) && \text{if}\quad (M+m)^2<u<(M+2m)^2\,,\\
&\varrho_{13}(s,t)=\varrho_{23}(s,t)=\beta_{13}(s,t)=\beta_{23}(s,t) && \text{if}\quad (M+m)^2<s<(M+2m)^2\,.
\end{aligned}$$

We define the elastic part of the meson-nucleon scattering amplitude by the Mandelstam representation with the spectral functions

$$\left\{\begin{aligned}
&\varrho_1^{\text{el}}(s)=\varrho_2^{\text{el}}(s)=\pi g^2\,\delta(s-M^2)+\theta\big(s-(M+m)^2\big)\beta_1(s)\,,\\
&\varrho_3^{\text{el}}(t)=0\,,\\
&\varrho_{12}^{\text{el}}(s,u)=\beta_{12}(s,u)+\beta_{21}(u,s)\,,\\
&\varrho_{13}^{\text{el}}(s,t)=\varrho_{23}^{\text{el}}(s,t)=\beta_{13}(s,t)=\beta_{23}(s,t)\,.
\end{aligned}\right. \tag{51}$$

Then

$$\begin{aligned}
&\varrho_1^{\text{el}}(s)=\varrho_1(s) \quad \text{if}\ (M+m)^2<s<(M+2m)^2\,,\\
&\varrho_{12}^{\text{el}}(s,u)=\varrho_{12}(s,u)\ \text{if}\ (M+m)^2<s<(M+2m)^2\ \text{or}\ (M+m)^2<u<(M+2m)^2\,,\\
&\varrho_{23}^{\text{el}}(s,t)=\varrho_{13}^{\text{el}}(s,t)=\varrho_{13}(s,t)=\varrho_{23}(s,t)\quad \text{if}\quad (M+m)^2<s<(M+2m)^2\,.
\end{aligned}$$

The scattering amplitude can therefore be written as

$$\Phi=\Phi_{\text{el}}+\Phi_{\text{inel}}\,,$$

where both terms satisfy the Mandelstam representation with symmetry in s and u. The spectral functions ϱ^{inel} of Φ_{inel}

$$\varrho_i^{\text{inel}}=\varrho_i-\varrho_i^{\text{el}}\,,\qquad \varrho_{ij}^{\text{inel}}=\varrho_{ij}-\varrho_{ij}^{\text{el}}$$

are crossing symmetric

$$\varrho_{12}^{\text{inel}}(s,u)=\varrho_{12}^{\text{inel}}(u,s)\,,\qquad \varrho_{13}^{\text{inel}}=\varrho_{23}^{\text{inel}}$$

and vanish outside the elastic regions of the meson-nucleon scattering,

$$(52)\qquad \begin{cases} \varrho_i^{\text{inel}}(s) = \varrho_2^{\text{inel}}(s) = 0 & \text{if} \quad s < (M+2m)^2\,, \\ \varrho_{12}^{\text{inel}}(s,u) = 0 & \text{if} \quad s < (M+2m)^2 \quad \text{or} \quad u < (M+2m)^2\,, \\ \varrho_{13}^{\text{inel}}(s,t) = \varrho_{23}^{\text{inel}}(s,t) = 0 & \text{if} \quad s < (M+2m)^2\,. \end{cases}$$

Hence Φ_{inel} is analytic in s and t except for the cuts

$$s \geqslant (M+2m)^2\,, \qquad u \geqslant (M+2m)^2\,, \qquad t \geqslant 4m^2\,.$$

For Φ_{inel} given three coupled non-linear integral equations follow for $\beta_1 = \beta_2$, $\beta_{12} = \beta_{21}$ and $\beta_{13} = \beta_{23}$. These are the eq. (49), where C, A_2, A_3 and T_0 must be substituted by the formulae

$$(49')\qquad C(s) = \frac{g^2}{M^2 - s - i\varepsilon} + \frac{1}{\pi}\int \mathrm{d}s'\, \frac{\beta_1(s') + \varrho_i^{\text{inel}}(s')}{s' - s - i\varepsilon}\,, \qquad \varepsilon \to +0\,,$$

$$(49'')\qquad A_2(u,s) = \pi g^2\,\delta(u - M^2) + \beta_1(u) + \varrho_1^{\text{inel}}(u) + \\ + \frac{1}{\pi}\int \mathrm{d}s'\, \frac{\beta_{12}(s',u) + \beta_{12}(u,s') + \varrho_{12}^{\text{inel}}(s',u)}{s' - s} + \frac{1}{\pi}\int \mathrm{d}t'\, \frac{\beta_{13}(u,t') + \varrho_{13}^{\text{inel}}(u,t')}{t' - t}\,,$$

$$(49''')\qquad A_3(t,s) = \varrho_3^{\text{inel}}(t) + \frac{1}{\pi}\int \mathrm{d}s'\, \frac{\beta_{13}(t,s') + \varrho_{13}^{\text{inel}}(t,s')}{s' - s} + \\ + \frac{1}{\pi}\int \mathrm{d}u'\, \frac{\beta_{13}(t,u') + \varrho_{13}^{\text{inel}}(t,u')}{u' - u}\,,$$

$$(49'''')\qquad T_0(s) = C(s) + \frac{1}{2\pi}\int \mathrm{d}\zeta\, A_3(t(\zeta), s + i\varepsilon)\, \lg \frac{\zeta + 1}{\zeta - 1} - \\ - \frac{1}{2\pi}\int \mathrm{d}\zeta\, A_2(u(\zeta), s + i\varepsilon)\, \lg \frac{\zeta + 1}{\zeta - 1}\,.$$

If the β are solutions of (50) (for ϱ_{inel} crossing symmetric and vanishing in (52)) the amplitude Φ, constructed by (41), (51) and $\varrho = \varrho^{\text{el}} + \varrho^{\text{inel}}$, is crossing symmetric and satisfies the unitarity condition for $(M+m)^2 \leqslant s < (M+2m)^2$.

If in (49) the inelastic spectral functions ϱ^{inel} are set equal to zero one obtains Wilson's integral equations for meson-nucleon scattering ([1,2]). This system de-

scribes the crossing symmetric one-meson approximation which has been formulated by MANDELSTAM in perturbation theory (M1) ([12]).

* * *

The author wishes to thank Dr. J. R. OPPENHEIMER for the hospitality extended to him at the Institute for Advanced Study, and the National Science Foundation for financial support. He is indebted to several physicists at this Institute for valuable discussions.

([12]) The iteration solution of this approximation scheme satisfies

$$\beta_{13}(s, t) = \beta_{23}(s, t) = 0 \qquad \text{for} \qquad t < 4M^2 \;\; (\text{instead of } t < 16m^2)\,,$$

the elastic form of the unitarity condition is valid up to $s = 9M^2$ in agreement with the rules found by Mandelstam in perturbation theory.

RIASSUNTO (*)

I contributi alla rappresentazione di Mandelstam dell'ampiezza di scattering dovuti a due particelle vengono separati simultaneamente per i tre canali di reazione.

(*) *Traduzione a cura della Redazione.*

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On the « Quasi-Elastic Diffraction » Scattering of High-Energy Protons.

B. T. Feld (*) and Chikashi Iso (**)

CERN - Geneva

(ricevuto il 27 Marzo 1961)

Summary. — The observations of Cocconi *et al.*, on the « quasi-elastic diffraction » scattering of (10÷25) GeV/c protons by nucleons, are explained as resulting from « isobar » excitation of the target nucleons. The experiments are shown to provide evidence on the excitation of the first three levels. Some physical arguments are presented concerning the nature of the excitation process and its energy and angular dependence.

Introduction.

Cocconi, Diddens, Lillethun and Wetherell (1) have observed, in the spectrum of protons scattered by free nucleons through small angles, a peak of inelastically scattered protons of momentum ~ 1 GeV/c less than that of the elastically scattered protons. This peak appears to be characterized by a « diffraction » angular distribution, of angular spread roughly comparable to that of the elastic « shadow » scattering. The most striking feature of the phenomenon, however, is the apparent constant energy difference between the elastic and the inelastic peaks, independent of the incident proton momentum (which was varied in the experiments between 9 and 25 GeV/c) and of the scattering angle (between 20 and 60 milliradians in the lab. system).

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(**) On leave from Tokyo University of Education, Tokyo, Japan.

(1) G. Cocconi, A. N. Diddens, E. Lillethun and A. M. Wetherell: *Phys. Rev. Lett.*, **6**, 231 (1961).

We present, below, some arguments in favour of an interpretation of these observations as resulting from processes in which the target nucleon is left in a definite excited state, or states, the ones corresponding to the $D_{\frac{3}{2}}$ or $F_{\frac{5}{2}}$ pion-nucleon resonances, of isotopic spin $\frac{1}{2}$, observed with pions of kinetic energy ~ 600 and ~ 900 MeV [2]. The arguments are based primarily on kinematical considerations. But, in Section **3**, we attempt to give a physical justification of our interpretation and discuss the circumstances which would favour the excitation, in such processes, of the various known nucleon « isobars ».

1. – Kinematical description.

Van Hove [3] has pointed out that the observations can be described by the kinematical properties of the scattering process represented in Fig. 1. M^* represents an « excited » nucleon, of excitation energy Δ (or rest-mass

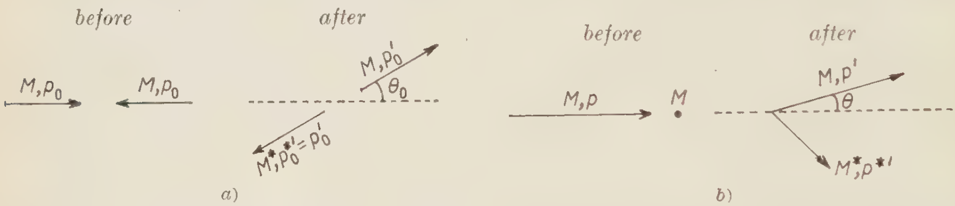

Fig. 1. – *a*) Kinematics in the center of mass system (c.m.s.); *b*) kinematics in the laboratory system (l.s.). Note that the subscript « 0 » stands for c.m.s.

$M^* = M + \Delta$). Consider an incident proton of total energy [*] u. In this case, when projectile and target masses are equal (and equal to 1), we have for the usual quantities describing the c.m.s. motion

$$\gamma_0 = u_0 = (1 - \beta_0^2)^{-\frac{1}{2}} = [(u+1)/2]^{\frac{1}{2}} , \tag{1}$$

$$\beta_0 \gamma_0 = p_0 = [(u-1)/2]^{\frac{1}{2}} . \tag{2}$$

Consider, first, elastic scattering ($\Delta = 0$, $M = M^*$) through the *small* angle θ_0.

(2) See, for example, P. Falk-Vairant and G. Valladas: *Proceedings Conference on High Energy Physics*, (1960), University of Rochester, p. 38.

(3) L. Van Hove: private communication; see also ref. (1).

(*) Unless specifically stated to the contrary, all masses are given in units of the proton mass M, all momenta in units of Mc and all energy in units of $Mc^2 = 0.938$ GeV. As usual $c = \hbar = 1$.

Then, in the small angle approximation,

$$\theta = \theta_0/2\gamma_0\,, \tag{3}$$

$$(p-p')/p = \Delta p_1/p = \frac{1}{4}\left(\frac{u}{u+1}\right)\theta_0^2 = \frac{1}{2}u\theta^2\,. \tag{4}$$

In Table I we have tabulated the values of Δp_1 observed by COCCONI *et al.* [1] as well as the values computed from eq. (4).

TABLE I. – *Observations of* COCCONI *et al.* [1] *on the positions of the elastic and inelastic peaks.*

p (GeV/c)	θ (mrad)	Δp_1 (GeV/c)	Δp_2 (GeV/c)	$\Delta p_{\text{elastic}}^{(\text{calc.})}$ (GeV/c)
8.95	60	0.35 ± 0.2	1.0 ± 0.1	0.16
12.1	60	0.5 ± 0.2	1.0 ± 0.1	0.28
15.2	60	0.7 ± 0.2	1.0 ± 0.1	0.44
16.0	40	~ 0.2	1.0 ± 0.1	0.22
16.2	60	0.7 ± 0.2	0.9 ± 0.1	0.50
17.6	60	0.7 ± 0.2	1.0 ± 0.1	0.60
18.6	60	0.7 ± 0.2	1.0 ± 0.1	0.66
19.5	60	0.8 ± 0.2	1.0 ± 0.2	0.73
21.9	60	0.8 ± 0.3	~ 1	0.92
24.2	20	~ 0.3	0.8 ± 0.1	0.13
24.2	40	~ 0.2	1.0 ± 0.1	0.53

Now, we consider the inelastic scattering process depicted in Fig. 1. For fixed Δ ($M^* = 1 + \Delta$), the final c.m.s. momenta $p_0' = p_0 - \delta_0$ are determined entirely by conservation of energy and momentum, and are given (for $\delta_0 \ll p_0$) by (*)

$$\delta_0 = \frac{\Delta(1+\Delta/2)}{2p_0} = \frac{\Delta'}{2p_0}\,. \tag{5}$$

Corresponding to this c.m.s. momentum change, we may compute the momentum shift of the line corresponding to the inelastically scattered projectile

(*) The exact formula is

$$\delta_0 = \frac{\Delta'}{2p_0}\left[1 - \frac{\Delta'}{2(u+1)}\right]. \tag{5'}$$

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in the small angle approximation

$$(6a)\qquad p - p' \equiv \Delta p_1 + \Delta p_2 ,$$

$$(6b)\qquad \Delta p_2(0) = \Delta p_2(0)\left(1 - \frac{u_0}{u}\Delta p_1\right),$$

$$(6c)\qquad \Delta p_2(0) = \frac{u}{u_0}\,\delta_0 \simeq \frac{u}{p}\,\Delta' .$$

We observe from eq. (6) that to a very good approximation, at the energies and angles covered by the experiments under consideration, constant Δp_2 implies constant target nucleon excitation Δ (Δ'). To obtain the observed values of $\Delta p_2 \simeq 1.0$, see Table I, we require

$$\Delta \simeq 0.7 \pm 0.1 .$$

The width of the inelastic peak will be determined both by the experimental resolution, which is what should determine the width of the elastic peak, and by the natural spread in the excitation energies of the excited target (*). Let the natural width of the excited nucleon state (M^*) be Γ_0. Differentiation of eq. (6c) gives, for the momentum spread in the l.s. of the inelastic peak

$$(7)\qquad \Gamma \simeq \frac{u}{p}\,(1+\Delta)\,\Gamma_0 = \frac{u}{p}\left(\frac{M^*}{M}\right)\Gamma_0 .$$

Actually, owing to the large (non-resonant inelastic) background and the finite resolution of the experimental observations, it is not possible to quote an experimental value of Γ. While it appears that the observed Γ is somewhat larger than the instrumental resolution, its evaluation must await improved experiments.

(*) Note that the process under consideration corresponds to the excitation of the target nucleon, with the projectile remaining unaltered and continuing forward in the c.m.s. Since the situation is symmetrical in the c.m.s., it should also be possible to excite the forward scattered nucleon. However, in this case, owing to the subsequent rapid decay

$$M^* \to M + n\pi ,$$

the inelastic protons will be spread over a broad range of l.s. momenta, and will be lost in the general background of inelastically scattered protons.

(4) A. M. WETHERELL: *Report Presented at the Conference on Strong Interactions* (Berkeley, Calif., Dec. 27-29, 1960), *Rev. Mod. Phys.* (in press).

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2. – Isobar interpretation.

Both the constancy of Δp_2 (therefore of Δ) and the apparent narrow width of the inelastic peaks suggest that we may be dealing, in these observations, with the excitation of a nucleon « isobar », possibly one of the levels which give rise to the peaks in the pion-nucleon cross-section [2]. This possibility has been previously suggested by WETHERELL [4].

In Table II we list the properties of the observed levels, together with the expected values of Δp_2 and Γ corresponding to their excitation by nucleons of initial l.s. energy $\geqslant 10$ GeV.

TABLE II. – *Properties of nucleon « isobars » and their « quasi-elastic » excitation.*

Description	Isotopic spin	Angular momentum	Pion l.s. kinetic energy at resonance (MeV)	Pion c.m.s. kinetic energy (MeV)	Δ	Γ_0 (MeV)	$\Delta p_2(c)$ (GeV/c)	Γ (MeV)
(3, 3)	$\frac{3}{2}$	$\frac{3}{2}^+$	200	160	0.32	105	0.35	139
(1, 3)	$\frac{1}{2}$	$\frac{3}{2}^-$	600	440	0.62	130	0.76	210
(1, 5)	$\frac{1}{2}$	$\frac{5}{2}^-$	900	615	0.80	115	1.06	208
(3, 1)	$\frac{3}{2}$	$\geqslant \frac{5}{2}$	1 350	835	1.04	~ 200	1.48	~ 410

Of the levels listed, only the second (1,3) and third (1,5) are candidates for explaining the observed quasi-elastic peak. The experiments [1] are, however, not inconsistent with an unresolved superposition of peaks from the two levels, both from the point of view of the position and the observed widths.

It remains, however, to understand the apparent absence of peaks corresponding to the other resonances. In the case of the lowest (3,3) state, it cannot yet be excluded experimentally that a peak corresponding to its excitation may be « hidden » in the observed elastic peak. In fact, comparison of the observed and computed values of Δp_1, in Table I, lends some evidence to this possibility, especially at the lower energies, since the observed values of Δp_1 are consistently greater than the computed $\Delta p_{\text{elastic}}$. This could be due to a shifting of the elastic peak by unresolved combination with the (3,3) inelastic peak, but it could also be due to experimental errors in the determination of Δp_1.

However, further weight is given to the possibility of « quasi-elastic » excitation of the (3,3) isobar, especially at low incident energies, by the obser-

vations of the Brookhaven group (⁵) on the spectra of protons, of initial (total) energies between ~ 2 and 2.5 GeV, scattered at the l.s. angle of $\sim 5°$. Clear indications were found for the excitation of the (3, 3) isobar, this time resolved (but, apparently, not completely) from the elastic peak. (Values of $\Delta \simeq 0.2$ were derived from the observations). The Brookhaven group also looked for indications of excitation of the higher resonances; but, although the available energies were sufficient for excitation of the next two resonances, they found no indication of peaks corresponding to their excitation.

As for the highest (3, ?) isobar, there have so far been no evidences of its excitation, although it could possibly be lost in the background of inelastically scattered protons observed in the measurements of COCCONI *et al.* [1].

3. – Physical considerations.

Diffraction (or shadow) elastic scattering is generally described as a purely optical phenomenon accompanying the absorption, within a radius R, of a beam of particles of wavelength $\lambda \ll R$. The characteristic scattering angle is $\theta_0 \simeq \lambda/R = (kR)^{-1}$. Alternatively, the process may be characterized by the transverse transfer of momentum $(\hbar/R)$, presumably through the exchange of mesons, as indicated schematically in Fig. 2*a*.

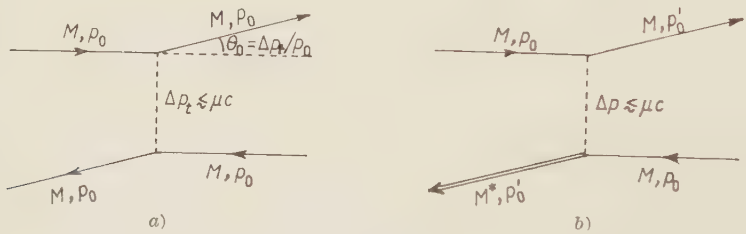

Fig. 2. – *a*) Diagram corresponding to diffraction elastic scattering. *b*) Diagram corresponding to « quasi-elastic » diffraction scattering.

The diagram corresponding to the « quasi-elastic » diffraction scattering, Fig. 2*b*, differs from that for elastic scattering only in that the target nucleon emerges from the « exchange » in an excited state; the momentum transfer is still $\sim \mu c$, but this time there must be a longitudinal component to permit conservation of energy and momentum in the collision.

Although it is possible that the properties of such collisions might be ana-

(⁵) G. B. CHADWICK, G. B. COLLINS, C. E. SWARTZ, A. ROBERTS, S. DE BENEDETTI, N. C. HIEN and P. J. DUKE: *Phys. Rev. Lett.*, 4, 611 (1960).

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lysed by means of the dispersion approach ([6]), we have preferred to consider them from a somewhat more classical, albeit quite qualitative point of view. We regard the nucleon as a « complex » system, with energy levels separated from the ground state by energies Δ_i. If a time varying perturbation is applied to such a system then, quite generally, those levels will be excited most strongly for which the frequency of the perturbing force is « in resonance with » the level, $\omega_i \simeq \Delta_i$.

Now, consider a peripheral collision, at an impact parameter $R \simeq \mu^{-1}$. In the c.m.s., the collision corresponds to an impulse of duration

$$\tau_0 \simeq R/\beta_0\gamma_0 \,. \tag{8}$$

Such an impact may be characterized by a relatively flat Fourier spectrum of frequencies up to (*) $\omega_0 \approx 2/\tau_0$. Accordingly, those levels will be most strongly excited for which

$$\Delta_i \leqslant \Delta_0 \simeq \frac{2}{\tau_0} \simeq 2\beta_0\gamma_0\mu = 2\mu p_0 \,. \tag{9}$$

Furthermore, the longitudinal momentum transfer in such collisions is, from eq. (5)

$$\delta_0 \leqslant \frac{\Delta_0}{2p_0} \approx \mu \,, \tag{10}$$

in justification of our earlier statement concerning the small momentum transfers in Fig. 2.

An additional consequence of this approach is that it suggests the possibility of a limitation, also on the change of *internal* angular momentum of the target in the excitation process (**). Thus, assuming that it remains appropriate to view these processes in the c.m.s. (Fig. 2), the possible internal angular momentum transfer in the impact is

$$\Delta J \leqslant R\delta_0^{\max} \approx 1 \,. \tag{11}$$

([6]) *E.g.*, S. D. Drell: *Peripheral Contributions of High-Energy Interaction Processes*, presented at the Conference on Strong Interactions (Berkeley, Calif., Dec. 27-29, 1960), *Rev. Mod. Phys.* (in press).

(*) For example, a « step » impulse of duration τ_0 has the Fourier spectrum $J_1(\omega\tau_0)/\omega\tau_0$, which falls to half the initial value at $\omega_0 \simeq 2.2/\tau_0$, and to zero at $\omega_0 \simeq 3.8/\tau_0$.

(**) Note that it is important, for these considerations, that the internal state of the projectile nucleon be unchanged. Whether this applies also to its charge is a problem whose discussion we postpone until the next section.

Clearly, the details depend on the nature of the nucleonic system and of the impacts; but, in general, those levels will tend to be excited most strongly for which the relations (9)–(11) tend to be obeyed.

Table III lists values of Δ_0 (eq. (9)) for a number of incident proton energies, and also lists the required longitudinal momentum transfers for the excitation of the first four nucleonic isobars (eq. (5)). It is clear from these numbers

TABLE III. – *Maximum excitation energies and required transverse momentum transfers for isobar excitation.*

u	$\Delta_0=2\mu p_0$	δ_0/μ (3, 3)	δ_0/μ (1, 3)	δ_0/μ (1, 5)	δ_0/μ (3. ?)
2	0.21	1.66	3.34	4.37	5.55
5	0.42	0.86	1.80	2.44	3.27
10	0.63	0.58	1.24	1.71	2.33
15	0.78	0.47	1.00	1.39	1.91
20	0.91	0.40	0.87	1.21	1.67
25	1.03	0.36	0.77	1.07	1.49
50	1.47	0.25	0.55	0.76	1.06

that collision in the $(1\div 5)$ GeV range favour excitation of the (3, 3) level only, in agreement with the Brookhaven observations [5]; that excitation becomes favourable for the next two, (1, 3) and (1, 5), levels in the $(10\div 25)$ GeV range [1]; our angular momentum considerations, insofar as they are important, favour the (1, 3) level excitation over both the (3, 3) and the (1, 5), especially at the upper end of this energy range. Finally, since the maximum frequency in the impact, Δ_0, as well as the momentum transfer δ_0, increase only as $u^{\frac{1}{2}}$, it will be necessary to increase the incident energy quite considerably in order to produce conditions favourable for the excitation of still higher isobar levels.

4. – Summary. Conclusions and discussion.

We have shown that the observations of COCCONI *et al.* [1] on the position of the elastic and « quasi-elastic » peaks in p-N scattering are kinematically consistent with a diffraction scattering mechanism, in which the target nucleon can be excited into a low-lying « isobar » level as well as left in its ground state. Some physical considerations have been presented which, although they cannot be considered as a « derivation » of the effect, indicate the plausibility of the excitation mechanism assumed to explain the observations and give rise to certain kinematical restrictions on the levels it is possible to excite as a function of the projectile energy.

Our description has certain features in common with the « diffraction dis-

sociation » mechanism, discussed by a number of authors (7) and suggested by VAN HOVE (3) as the source of the quasi-elastic scattering process. However, at least as most commonly interpreted, the diffraction dissociation process cannot lead to a change in any of the internal quantum numbers of the system-charge, spin, isotopic spin, strangeness, etc.; if this were true, none of the isobar levels could be excited. On the contrary, our mechanism permits changes in the quantum numbers, although it does suggest certain restrictions on the possible changes.

In particular, our model gives rise to a number of further consequences, some of which can be tested experimentally:

1) Since the isotopic spin of the excited nucleon is fixed, depending on the level excited, the charge of the scattered nucleon is also determined, Thus, in p-p collisions leading to the quasi-elastic peak of COCCONI *et al.*, the scattered nucleon must always be a proton, since the levels which could be responsible for this peak both have $t=\frac{1}{2}$. For the excitation of the (3, 3) level, on the other hand, the final state must have the definite ratio of 1:3 between protons and neutrons, since the total isotopic spin of the system is fixed ($t=1$, $t_3=1$).

2) For p-n collisions, charge-exchange excitation becomes quite possible, even for levels with $t=\frac{1}{2}$. However, it is not possible, to make an *a priori*, prediction of the proton to neutron ratio in the final state (*), since this depends on the relative magnitude and phase of the amplitudes corresponding to the two values of the total isotopic spin (0 and 1) present in the initial state. For excitation of the (3, 3) level, however, the charge prediction again becomes unique, and is 1:1 for the final proton-to-neutron ratio.

3) Of course, the pion and nucleon charge and momentum distributions from the decay of the excited target should also be predictable. Their observation, however, will require visual techniques, such as bubble chambers or emulsions.

4) For the case of incident π- or K-mesons, instead of protons, the same type of quasi-elastic scattering processes should be observed, with Δp_1, Δp_2, and Γ given by eq. (4), (6) and (7), respectively, to a good approximation; for these quantities depend mainly on the mass and excitation of the target nucleon, and only very weakly on the mass of the incident particle, at least in the energy range of interest to us. The observation of such quasi-elastic scattering for mesons would provide an excellent test of this model, as compared to other

(7) E. L. FEINBERG and J. I. POMERANČUK: *Suppl. Nuovo Cimento*, **3**, 652 (1956); M. L. GOOD and W. D. WALKER: *Phys. Rev.*, **120**, 1857 (1960).

models, such as one which describes the effect to the diffraction scattering of the nucleon by a pion in the field of the target (8); or one in which the effect is due to the exchange of a single π^0 (1).

Similar arguments, but based on a perturbation field-theoretic approach, were applied by SELOVE (9) in discussing the Brookhaven results (5). We are grateful to A. ROBERTS and G. B. CHADWICK for recalling this work to our attention.

It has also been pointed out to us by a number of our colleagues, and most persuasively by J. S. BELL, that since, when viewed in the c.m.s. the levels of the target nucleon will suffer a « red shift », $\Delta_i \to \Delta_i/\gamma_0$ the maximum excitation energy should be $\gamma_0\Delta_0$ rather than Δ_0 (see eq. (9) and Table III column 2). Correspondingly, it is not clear whether the maximum angular momentum transfer should be taken as ~ 1 (eq. (11)) or as $\sim\gamma_0$. Eq. (10), for the momentum transfer, holds in any case. These considerations weaken the conclusions drawn at the end of Section 3 insofar as the CERN experiments are concerned, in the direction of enhancing the expectation for observing the excitation of all the known nucleon isobars in this energy range. But they do not appreciably alter our conclusions with respect to the Brookhaven results.

* * *

We are grateful to W. M. LAYSON, S. OKUBO and to other members of the CERN theoretical group, for stimulating discussions and searching criticisms.

(*) It should be noted that the same type of argument might also be applied to the diffraction elastic scattering, since it is also possible that the amplitudes for this type of scattering differ in the two isotopic spin states.

(8) D. AMATI: private communication.

(9) SELOVE: *Phys. Rev. Lett.*, 5, 163 (1960).

RIASSUNTO (*)

Le osservazioni di COCCONI *et al.*, sullo scattering della « diffrazione quasi-elastica » dei protoni di (10 ÷ 25) GeV/c su nucleoni, vengono spiegate come il risultato di una eccitazione « isobara » dei nucleoni del bersaglio. Mostriamo che gli esperimenti forniscono prove della eccitazione dei primi. Presentiamo alcuni argomenti fisici concernenti la natura del processo di eccitazione e la sua dipendenza dall'energia e dall'angolo.

(*) *Traduzione a cura della Redazione.*

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The Eriksen Transformation in the Case of Particles with Spin 0 and 1.

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(ricevuto il 27 Marzo 1961)

Summary. — It is shown how Eriksens transformation can be used in the reduced Kemmer theory, and how this simplifies scattering and spin precession of the vector particle.

The formal procedure of obtaining the Foldy-Wouthuysen (1) representation for fermions has recently been improved in the articles by Eriksen (2) and Eriksen and Kolsrud (3). In principle the Eriksen and Kolsrud transformation gives in closed form a transformation to two-component theory with separated positive energies in the general case of time dependent external fields.

Their transformation methods can also be extended to incorporate the spin 0 and spin 1 case as is easily shown. (We discuss for brevity only the case of time independent fields).

Particles with rest mass $\neq 0$ can be described by the Kemmer equation

$$\{\beta_\mu \partial_\mu^{(-)} + m\}\psi = 0 \,,$$

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \delta_{\mu\nu}\beta_\lambda + \delta_{\nu\lambda}\beta_\mu \,, \qquad \partial_\mu^{(-)} = \frac{\partial}{\partial x_\mu}(-ieA_\mu) \,,$$

(1) L. L. Foldy and S. A. Wouthuysen: *Phys. Rev.*, **78**, 23 (1950).
(2) E. Eriksen: *Phys. Rev.*, **111**, 1011 (1958).
(3) E. Eriksen and M. Kolsrud: *Suppl. Nuovo Cimento*, **18**, 1 (1960).

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or by the physically equivalent Hamiltonian equation ([4])

$$H\psi = i\,\frac{\partial\psi}{\partial t}\,,$$

$$H = \frac{1}{2m}\,\tau_3(1-\tau_2)\{(\boldsymbol{p}-e\boldsymbol{A})^2 - e\boldsymbol{S}\boldsymbol{H})\} - \frac{i}{m}\,\tau_1\{\boldsymbol{S}(\boldsymbol{p}-e\boldsymbol{A})\}^2 + \tau_3 m + e\varphi\,,$$

where

$$\tau_1 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}, \qquad \tau_2 = \begin{Bmatrix} 0 & -i \\ i & 0 \end{Bmatrix}, \qquad \tau_3 = \begin{Bmatrix} 1 & 0 \\ 0 & -1 \end{Bmatrix},$$

and $\{S_i\}$ are the spin matrices for the particle in question

$$s = 0: \qquad S_i = 0\,, \qquad s = 1:$$

$$S_x = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{Bmatrix}, \qquad S_y = \begin{Bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{Bmatrix}, \qquad S_z = \begin{Bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}.$$

The wave functions are in this case normalized according to the metric

$$\int \psi^* \tau_3 \psi = \pm 1\,.$$

In the free-particle case:

$$H = H_0 = \frac{1}{2m}\,\tau_3(1-\tau_2)\boldsymbol{p}^2 - \frac{i}{m}\,\tau_1(\boldsymbol{S}\boldsymbol{p})^2 + \tau_3 m\,, \qquad H_0^2 = p^2 + m^2\,,$$

$$(\text{note } (\boldsymbol{S}\boldsymbol{p})^4 = p^2(\boldsymbol{S}\boldsymbol{p})^2),$$

that is — we have eigenvalues $\pm E$.

By looking at the free particle solutions (which in the p representation simply are the columns of $H\tau_3 + E$) we see that solutions corresponding to positive and negative energy eigenvalues are charge-conjugate.

Therefore, by transforming H to an even form, and $\lambda \equiv U/\sqrt{H^2}$ to τ_3 simultaneously, we can describe the two charges separately,

$$\frac{1}{2}(1 \pm \lambda) \to \frac{1}{2}(1 \pm \tau_3)\,, \qquad H^T = \left(\begin{array}{c|c} & 0 \\ \hline 0 & \end{array}\right).$$

([4]) M. Taketani and S. Sakata: *Proc. Phys. Math. Soc.* (*Japan*), **22**, 757 (1939).

In the $s=\frac{1}{2}$ case the Eriksen-transformation (3)

$$U=(1+\beta\lambda)[2+\beta\lambda+\lambda\beta]^{-\frac{1}{2}}$$

has the property:

$$U\lambda U^\dagger=\beta\,,\qquad [UHU^\dagger,\beta]=0$$

and in the $s=0$ and $s=1$ case we simply replace β with τ_3.

$$U=(1+\tau_3\lambda)[2+\tau_3\lambda\tau_3]^{-\frac{1}{2}}$$

is unitary with respect to the metric $(\psi|\tau_3\psi)$

$$H^\dagger\equiv\tau_3\bar{H}\tau_3\,,$$

where $\bar{H}$ is the adjoint of H,

$$U^\dagger=\tau_3\bar{U}\tau_3=(1+\lambda\tau_3)[2+\tau_3\lambda+\lambda\tau_3]^{-\frac{1}{2}}=U^{-1}\,.$$

Further

$$[UHU^\dagger,\tau_3]=0\,,\qquad U\lambda U^\dagger=\tau_3\,.$$

In the case of general external fields we must evaluate $H^\dagger=UHU^\dagger$ in powers of l or $1/m$; however in the case where $[\tau_3,H^2]=0$ we get a very simple expression for H^T:

$$H(2+\tau_3\lambda+\lambda\tau_3)=\left(2H+H\tau_3\frac{H}{\sqrt{H^2}}+\sqrt{H^2}\tau_3\right)=(2+\lambda\tau_3+\tau_3\lambda)H\,,$$

We have:

$$):\quad [H,[2+\tau_3\lambda+\lambda\tau_3]^{-\frac{1}{2}}]=0\,,$$

further:

$$(1+\tau_3\lambda)H=H+\tau_3\sqrt{H^2}=H(1+\lambda\tau_3)\qquad):\quad UH=H^\dagger U$$

and

$$H^T=UHU^\dagger=U^2H=\tau_3\lambda H=\tau\sqrt{H^2}\,.$$

In the free particle case:

$s=0\quad H^T=\tau_3E\,,\quad H^T\psi^T=i\dfrac{\partial\psi^T}{\partial t}$ is equal to two 1-component equations ,

$s=\frac{1}{2}\quad H^T=\beta E\,,\quad H^T\psi^T=i\dfrac{\partial\psi^T}{\partial t}$ is equal to two 2-component equations ,

$s=1\quad H^T=\tau_3E\,,\quad H^T\psi^T=i\dfrac{\partial\psi^T}{\partial t}$ is equal to two 3-component equations .

In the case of a homogeneous magnetic field

$$s=0 \qquad H^T = \tau_3[(\boldsymbol{p}-e\boldsymbol{A})^2+m^2]^{\frac{1}{2}} \equiv \tau_3 E_\pi\,,$$

$$s=\tfrac{1}{2} \qquad H^T = \beta[E_\pi^2 - e\boldsymbol{\sigma}\boldsymbol{H}]^{\frac{1}{2}}\,,$$

$$s=1 \qquad H^T = \tau_3\left|E_T^2 - e\boldsymbol{S}\boldsymbol{H}\left\{1+\frac{1}{m^2}(\boldsymbol{S}\boldsymbol{\pi})^2\right\}\right|^{-\frac{1}{2}}, \qquad \text{where } \pi \equiv \boldsymbol{p}-e\boldsymbol{A}.$$

): to 1. order in e.

$$s=\tfrac{1}{2} \qquad H^T = \beta E_\pi\left\{1-\frac{e}{2E_\pi^2}\,\boldsymbol{\sigma}\boldsymbol{H}\right\},$$

$$s=1 \qquad H^T = \tau_3 E_\pi\left\{1-\frac{e}{2E_\pi^2}\,\boldsymbol{S}\boldsymbol{H}\left(1+\frac{1}{m}(\boldsymbol{S}\boldsymbol{\pi})^2\right)\right\}.$$

The precession of the spin in a magnetic field is given by $\mathrm{d}\boldsymbol{s}/\mathrm{d}t = i[H, \boldsymbol{s}]$ and for particles with a definite charge): τ_3 and β quantized

$$s=\tfrac{1}{2} \qquad \frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t} = \frac{e}{2E_\pi^2}\,\boldsymbol{\sigma}\times\boldsymbol{H}\,,$$

$$s=1 \qquad \frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \frac{e}{2E_\pi^2}\left\{\boldsymbol{S}\times\boldsymbol{H}\left(1+\frac{1}{m^2}(\boldsymbol{S}\boldsymbol{\pi})^2\right)+\frac{1}{m^2}\,\boldsymbol{S}\boldsymbol{H}(\{\boldsymbol{S}\boldsymbol{\pi}\}\{\boldsymbol{S}\times\boldsymbol{\pi}\}+\{\boldsymbol{S}\times\boldsymbol{\pi}\}\{\boldsymbol{S}\boldsymbol{\pi}\})\right\}.$$

This shows very clearly how much more complicated the behaviour of the vector meson is, when $\boldsymbol{S}\boldsymbol{\pi}=0$ the precession is normal, otherwise it is strongly energy-dependent.

An evaluation of H^T in powers of $(1/m)^n$ in the case of an external electromagnetic field has been given by CASE [5] by means of the Foldy-Wouthuysen method, and as Eriksen's transformation gives the same result up to the orders of interest, we do not wish to repeat this. It is however easy to obtain a transformed Hamiltonian expanded in powers of the coupling-constant, especially the transformation to first order in e is trivial.

From the result it is easy to obtain the formula of Laporte and Massey and Corben for scattering of vector-mesons so we will show this.

Transformation to 1 order in e:

We write $H = H_0 + eH'$ and transform this with the free particle transformation

$$H = (1+\tau_3\lambda_0)[2+\tau_3\lambda_0+\lambda_0\tau_3]^{-\frac{1}{2}} = \frac{1}{2\sqrt{Em}}\{E+m+\tau_2(E-m)Q\}\,,$$

[5] K. M. CASE: *Phys. Rev.*, **95**, 1323 (1954).

where

$$Q = 2\left(\frac{\boldsymbol{Sp}}{p}\right)^2 - 1\,, \qquad U_0 H_0 U_0^\dagger = \tau_3 E\,,$$

$$):\quad H_1^T = U_0 H U_0^\dagger = \tau_3 E + e U_0 H' U_0^\dagger \equiv \tau_3 E + e\mathscr{E} + e\mathscr{O}\,, \qquad [\tau_3, \mathscr{E}] = [\tau_3, \mathscr{O}]_+ = 0.$$

By means of a transformation $\exp[eS]$ $S^\dagger = -S$ we now reduce H_1^T for odd terms in e^2

$$H_2^T = \exp[eS] H_1^T \exp[-eS] = H_1^T + e[S, H_1^T] + \frac{e^2}{2!}[S, [S, H_1^T]] =$$

$$= \tau_3 E + e\mathscr{E} + e(\mathscr{O} + [S, \tau_3 E]) + e^2\,.$$

By the choice

$$S = \int_{-\infty}^{0} \exp[Ez](\tau_3 \mathscr{O}) \exp[Ez]\, \mathrm{d}z\,,$$

$$[S, \tau_3 E] = -\tau_3 [S, E]_+ = -\int_{-\infty}^{0} \exp[Ez] E\, \mathscr{O} + \mathscr{O} E) \exp[Ez]\, \mathrm{d}z = -$$

$$-\int_{-\infty}^{0} \mathrm{d}\{\exp[Ez]\, \mathscr{O} \exp[Ez]\} = -\,\partial\,,$$

$$):\quad H_2^T = \tau_3 E + e\mathscr{E} + e^2 \ldots$$

In the case of an electrostatic field: $H' = e\varphi$, τ_3 quantized to 1 we get

$$s = 0 \qquad H^T = E + \frac{e}{2}\left\{\sqrt{E}\varphi \frac{1}{\sqrt{E}} + \frac{1}{\sqrt{E}}\varphi\sqrt{E}\right\} + e^2 \ldots\,,$$

$$s = 1 \qquad H^T = E + \frac{e}{4}\left\{\frac{E+m}{\sqrt{Em}}\varphi\frac{E+m}{\sqrt{Em}} - Q\frac{E-m}{\sqrt{Em}}\varphi\frac{E-m}{\sqrt{Em}} Q\right\} + e^2\,.$$

For elastic scattering in the $S = 0$ case we find

$$\mathrm{d}\sigma \sim 1\langle p_f | e\mathscr{E} | p_i\rangle|^2 = e|\varphi_{if}|^2 \qquad \text{as expected.}$$

In the $S = 1$ case (three-component wave functions) we choose the direction of the incoming particle as z axis:

): the eigenstates of $\boldsymbol{Sp}/p$ are eigenstates of $S_z p_z/p_z$

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where S_z has eigenvectors

$$|S'_z = 1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\\ 0\end{bmatrix}, \qquad |S'_z = -1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\\ 0\end{bmatrix}, \qquad |S'_z = 0\rangle = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

After the scattering by an angle θ in the yz plane say, we have as eigenstates for $\boldsymbol{Sp}/p$

$$\left|\left(\frac{\boldsymbol{Sp}}{p}\right)' = 1\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\cos\theta\\ -i\sin\theta\end{bmatrix}\exp[i\boldsymbol{p}_f\boldsymbol{x}],$$

$$\left|\left(\frac{\boldsymbol{Sp}}{p}\right)' = -1\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\cos\theta\\ i\sin\theta\end{bmatrix}\exp[i\boldsymbol{p}_f\boldsymbol{x}],$$

$$\left|\left(\frac{\boldsymbol{Sp}}{p}\right)' = 0\right\rangle = \begin{bmatrix}0\\ \sin\theta\\ \cos\theta\end{bmatrix}\exp[i\boldsymbol{p}_f\boldsymbol{x}],$$

$$Q\left|\left(\frac{\boldsymbol{Sp}}{p}\right)' = \begin{matrix}+1\\ -1\\ 0\end{matrix}\right\rangle = \begin{matrix}+1\\ +1\\ -1\end{matrix}\left|\left(\frac{\boldsymbol{Sp}}{p}\right)' = \begin{matrix}+1\\ -1\\ 0\end{matrix}\right\rangle, \qquad Q = 2\left(\frac{\boldsymbol{Sp}}{p}\right)^2 - 1.$$

In the first Born approximation

$$d\sigma \sim |\langle p_f|e\mathscr{E}|p_i\rangle|^2 = e^2|\mathscr{E}_{if}|^2$$

we obviously get:

$$\mathscr{E}_{1\to 1} = \tfrac{1}{2}(1+\cos\theta)\varphi_{fi},$$

$$\mathscr{E}_{1\to -1} = \tfrac{1}{2}(1-\cos\theta)\varphi_{fi},$$

$$\mathscr{E}_{1\to 0} = \frac{-i}{\sqrt{2}}\sin\theta\,\frac{E^2+m^2}{2mE}\,\varphi_{fi}.$$

$$d\sigma_1 \sim e^2\left(1+\frac{1}{2}\,\frac{p^4}{4m^2E^2}\sin^2\theta\right)|\varphi_{fi}|^2,$$

and similarly $d\sigma_{-1} = d\sigma_1$ while

$$d\sigma_0 \sim e^2\left(1+\frac{p^4}{4m^2E^2}\sin^2\theta\right)|\varphi_{fi}|^2.$$

For unpolarized vector-mesons

$$\mathrm{d}\sigma = \frac{1}{3}\{\mathrm{d}\sigma_1 + \mathrm{d}\sigma_0 + \mathrm{d}\sigma_{-1}\} \sim e^2 \left\{1 + \frac{1}{6}\frac{p^4}{m^2E^2}\sin^2\theta\right\}|\varphi_{fi}|^2 =$$

$$= e^2\left\{1 + \frac{1}{6}\left(\frac{pv}{m}\right)^6 \sin^2\theta\right\}|\varphi_{fi}|^2 ,$$

which is the formula of Laporte and Massey and Corben.

(6) O. Laporte: *Phys. Rev.*, **54**, 9(5 (1938).
(7) J. T. Massey and H. C. Corben: *Proc. Cam. Phys. Soc.*, **35**, 463 (1939).

RIASSUNTO (*)

Mostro come la trasformazione di Eriksen può essere usata nella teoria di Kemmer ridotta, e come questo semplifichi lo scattering e la precessione di spin della particella vettoriale.

(*) *Traduzione a cura della Redazione.*

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IL NUOVO CIMENTO Vol. XXI, N. 1 1° Luglio 1961

A Method for the Measurement of the Mobility of Electric Charges in Liquids.

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(ricevuto il 30 Marzo 1961)

Summary. — A new technique for the measurement of the mobility of electric charges in liquids has been developed. One essentially measures the time of flight between a grid and the collecting electrode which is connected to the electrometer by a suitable filter. This method is of easy application and the experimental results are in excellent agreement with those obtained in liquid helium by other authors.

1. – Introduction.

The method for the measurement of mobility described in this paper was designed for use in liquid helium at temperatures below the λ-point but could easily be extended to measurements in other liquids.

As in the methods used by M. A. Biondi and M. Chanin (1) for noble gases, by Meyer and Reif (2) for liquid helium, and by J. A. Hornbeck (3) and A. Dahm, etc. (4) for the mobility respectively of electrons and of negative charges in helium, the physical quantity which is measured is the time employed by an electric charge to travel the distance between two electrodes (time of flight), under the action of an electric field. It is then easy to

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(1) M. A. Biondi and M. Chanin: *Phys. Rev.*, **94**, 910 (1954).

(2) L. Meyer and F. Reif: *Phys. Rev.*, **110**, 279 (1958).

(3) J. A. Hornbeck: *Phys. Rev.*, **83**, 375 (1951).

(4) A. Dahm, J. Levine, J. Penley and T. M. Sanders, Jr.: *Proceedings 7th International Conference on Low Temperature Physics* (Toronto, August 1960).

find the drift velocity of the charge and hence the mobility through the well known relation $v = \mu E$.

The present method was studied and adopted in this laboratory especially because of the simplicity of the experimental apparatus which nevertheless permits completely satisfactory and accurate measurements to be made. For reasons which shortly shall become clear this technique has been named the current null method.

The system used to measure the times of flight will be described here and briefly discussed. Some of the experimental data obtained will then be compared with the values which have been found using other techniques, both in this laboratory (5,6) and by MEYER and REIF (2).

2. – Description of the apparatus and method.

The apparatus, of which Fig. 1 is a schematic drawing, is substantially a triode with silver electrodes. The α-particles emitted by the Polonium 210 deposited on the first electrode ionize a thin layer of the helium. The electric

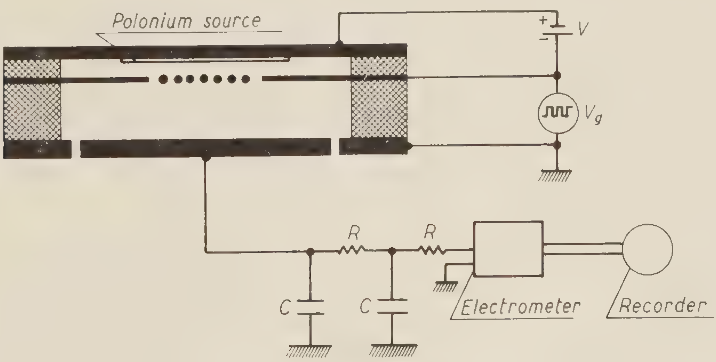

Fig. 1. – Schematic view of the experimental apparatus.

charges of a determined sign are then extracted and directed towards the grid by means of an electric field obtained by applying a constant potential difference between the radioactive electrode and the grid.

Between the grid and a third electrode (the collector electrode) an a.c. potential is maintained by means of a square wave generator.

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(5) G. CARERI, F. SCARAMUZZI and J. O. THOMSON: *Nuovo Cimento*, **13**, 186 (1959).

(6) G. CARERI, S. CUNSOLO and F. DUPRÉ: *Proceedings 7th International Conference on Low Temperature Physics* (Toronto, August 1960).

During the positive half-period the electric field drives the charges toward the collector, while during the negative half-period the field between grid and collector is reversed and the charges are turned back towards the grid.

Neglecting the effect of diffusion, which we can do within certain limits as we will see later on, at the end of the negative half-period of the applied voltage there will no longer be any electric charges in the space between the grid and the collector.

If we indicate by θ the half-period, and by ϱ the charge density, the quantity of charge arriving at the collector during the positive semi-period will be

$$\left\{\begin{array}{ll} q = \varrho\mu E(\theta - \tau) & \text{for } \theta \geqslant \tau\,, \\ q = 0 & \text{for } \theta \leqslant \tau\,, \end{array}\right. \tag{1}$$

where τ is the time taken for a charge to travel from the grid to the collector.

Since during the negative semi-period no charges arrive at the collector the average current will be

$$\left\{\begin{array}{ll} \bar{i} = \dfrac{i_0}{2}\left(1 - \dfrac{\tau}{\theta}\right), & \text{for } \theta \geqslant \tau, \\ \bar{i} = 0\,, & \text{for } \theta \leqslant \tau, \end{array}\right. \tag{2}$$

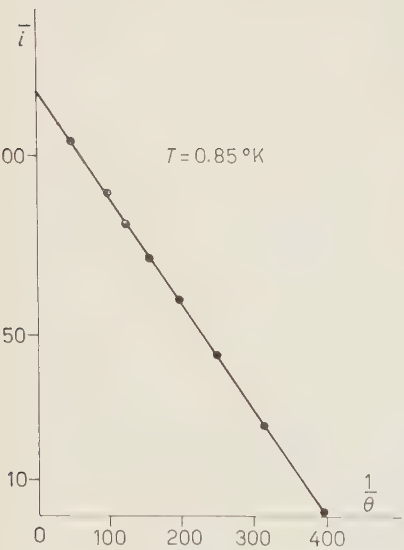

Fig. 2. – The average current received $\bar{i}$ plotted *vs.* the reciprocal of the time θ. This graph has been obtained for positive helium ions moving in liquid helium at 0.85 °K under an electric field of 8.75 volt/cm.

where $i_0 = \varrho\mu E$ is the value of the current which would be obtained applying a constant potential between the grid and the collector.

To measure the average value of the current a filter consisting of a resistor of $10^7\ \Omega$ and a capacitor of 0.1 μF was used to connect the collector to the electrometer. In this way the currents due to capacitive coupling were eliminated. Naturally the insulation resistance of the capacitors had to be much larger than the input resistor of the electrometer.

Plotting a graph of the average current as a function of $1/\theta$ one obtains a straight line which intersects the axis of the abscissa $1/\theta$ at the point $1/\theta = 1/\tau$. Such a graph, obtained with an electric field of 8.75 V/cm at a temperature of 0.85 °K is shown in Fig. 2. Once τ

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is known the mobility can be immediately calculated from the relation

$$\mu = \frac{s^2}{V_g \tau}\,, \tag{3}$$

where V_g is the absolute value of the potential difference between the grid and the collector and s is the distance between these two electrodes.

Square wave voltages V_g between 0 and 150 V are obtained by means of a flip-flop. This voltage can then be applied to a second flip-flop capable of furnishing square waves of amplitude as large as 600 V.

In this way one can make measurements in fields as high as 10^3 V/cm. The minimum value of the semi-period which can be reached with the present apparatus while maintaining a sufficiently good waveform is 10^{-5} s.

3. – Limits of the method.

One of the phenomena which must be taken into account in searching for limitations of the method of measurement described is the diffusion of the electric charges in the liquid.

Evaluating the ratio between the average current due to diffusion and the value, which the current would have if the grid were maintained at a constant potential with respect to the collector, one finds in the first approximation the following expression:

$$\frac{i_D}{i_0} = \frac{\sqrt{D}}{2\sqrt{\pi}\mu E\sqrt{\theta}}\left\{1 + \frac{1}{2}\exp\left[-\frac{(s-\mu E\theta)^2}{4D\theta}\right]\right\}, \tag{4}$$

where D is the coefficient of diffusion of the ion.

We note that the measurements are made in the current interval between $i_0/2$ and 0, while θ varies from values much greater than τ to values approximately egual to τ.

That which interests us is that i_D/i_0 be maintained lower, or at most equal, to the fractional error determined by the precision of measurement wanted. For this to happen we must satisfy the following relationship derived from (4) by use of the Einstein relation:

$$\frac{T}{V_g} \leqslant \varepsilon^2 \cdot 6.3 \cdot 10^4\,. \tag{5}$$

In the helium region the minimum values of the potential for which (5) can become restrictive are so low that normally one already finds an obstacle in the measurement of the current itself. At 1 °K, for example, we find

0.21 V as lower limit on the potential given by (5) for an i_D/i_0 of 1%, while a measurement of the current for a potential of 1 V is already rather difficult at this temperature.

An upper limit on the measurable velocity can be obtained from the maximum value of the frequency or potential reached by the square waves.

4. – Some experimental data.

We have examined and reported in Fig. 3 those measurements taken in the helium II region in electric field in which the mobility is found to be field-independent together with the values already obtained by other methods both in this laboratory (5,6) and by MEYER and REIF (7).

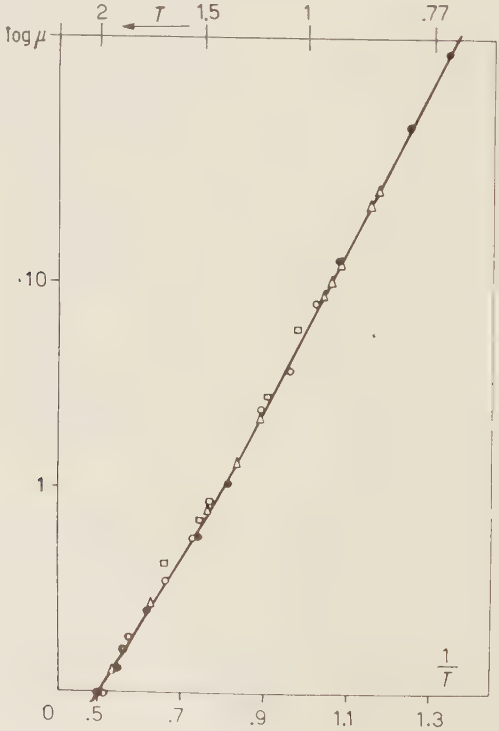

Fig. 3. – The experimental values of the mobility of positive ions in liquid helium obtained by different techniques *vs.* the reciprocal absolute temperature. The mobility units are cm²/volt s. ○ Heat flush c.s.t.; ● Shutters m.r.; □ space charge c.c.d.; △ current null method.

(7) F. REIF and L. MEYER: *Phys. Rev.*, 119, 1164 (1960).

The errors in the present measurements can be mostly attributed to errors in the temperature measurement.

For an interpretation of the results and for a picture of the physical problem refer to the article by CARERI [8].

Measurements of the mobility as a function of the electric field and of the pressure performed by this technique will appear in other papers.

APPENDIX

The influence of diffusion.

In order to see the influence of diffusion on the current measured, the problem can be outlined in the following manner. One supposes that before the time $t=0$ the electric charges are all situated between the grid and the radioactive electrode with a uniform density equal to ϱ_0. At time $t=0$ the charges pass the grid under the action of the electric field and diffusion. The equation which controls this phenomenon, supposing an infinite grid, is the following:

$$\frac{\partial^2 \varrho_+}{\partial x^2} - \frac{\mu E}{D}\frac{\partial \varrho_+}{\partial x} + \frac{1}{D}\frac{\partial \varrho_+}{\partial t} = 0\,. \tag{A.1}$$

If we leave out the action of the collector electrode the initial conditions are

$$\begin{cases} \varrho_+(x,\,0) = \varrho_0\,, & \text{for } x<0\,,\\ \varrho_+(x,\,0) = 0\,, & \text{for } x>0\,. \end{cases} \tag{A.2}$$

Since in the case of an unlimited domain the solutions of $(1-\theta)$ can be expressed through the relation

$$\varrho_+(x,\,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}\varrho_+(x_0,\,0)\exp\left[-\frac{(x-x_0-\mu Et)^2}{4Dt}\right]\mathrm{d}x_0\,, \tag{A.3}$$

the solution of (A.1) with the conditions (A.2) is

$$\varrho_+(x,\,t) = \frac{\varrho_0}{2} - \frac{\varrho_0}{\sqrt{\pi}}\,G\left(\frac{x-\mu Et}{4Dt}\right). \tag{A.4}$$

(8) G. CARERI: *Progress in Low Temperature Physics*, vol. III (to be published shortly), (Amsterdam).

where

$$G(y)=\int_0^y \exp\left[(-x^2)\,\mathrm{d}x\right].$$

At the end of the first half-period the electric field changes sign and therefore in place of (A.1) we must consider the equation:

$$\frac{\partial^2 \varrho_-}{\partial x^2}+\frac{\mu E}{D}\frac{\partial \varrho_-}{\partial x}-\frac{1}{D}\frac{\partial \varrho_-}{\partial t}=0\,. \tag{A.5}$$

If we neglect the effect of the electric charges on the opposite side of the grid and that of the grid itself, the initial conditions are now the following:

$$\begin{cases} \varrho_-(x,0)=0\,, & \text{for } x<0 \text{ and for } x>s\,,\\[2mm] \varrho_-(x,0)=\dfrac{\varrho_0}{2}-\dfrac{\varrho_0}{\sqrt{\pi}}\,G\left(\dfrac{x-\mu E\theta}{\sqrt{4D\theta}}\right), & \text{for } 0<x<s\,, \end{cases} \tag{A.6}$$

where s is the distance between grid and collector.

We now have the solution

$$\varrho_-(x,t)=\frac{\varrho_0}{2\sqrt{\pi}}\left[G\left(\frac{l-x-\mu Et}{\sqrt{4Dt}}\right)+G\left(\frac{x+\mu Et}{\sqrt{4Dt}}\right)\right]- \tag{A.7}$$

$$-\frac{\varrho_0}{\pi\sqrt{4Dt}}\int_0^l G\left(\frac{l-y_0-\mu E\theta}{\sqrt{4D\theta}}\right)\exp\left[-\frac{(l-x-y_0-\mu Et)^2}{4Dt}\right]\mathrm{d}y_0\,.$$

One observes that at the end of the first negative semi-period the $\varrho_-(x,t)$ is so small with respect to ϱ_0, that we can certainly maintain the validity of the conditions (A.2).

However it will in general be assured that (A.4) and (A.7) represent the electric charge distributions during the first semi-period in which the field is respectively positive or negative.

Knowing $\varrho_+(x,t)$ and $\varrho_-(x,t)$ it is easy to calculate the charge passed beyond the grid during the positive semi-period $\left(q_+=\int_0^\infty \varrho_+(x,t)\,\mathrm{d}x\right)$, that part of it which turns back during the negative semi-period $q_-=\int_{-\infty}^0 \varrho_-(x,\theta)\,\mathrm{d}x$, and therefore the ratio between the average current measured and the continuous value $\varrho_0\mu E$:

$$\frac{\bar{i}}{i_0}=\frac{q_+-q_-}{2\varrho_0\mu E\theta}\,.$$

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Thus one obtains to a very good approximation the following expression, valid when $2\sqrt{D\theta} \ll \mu E\theta$:

$$\frac{\bar{i}}{i_0} = \frac{1}{2}\left(1 - \frac{\tau}{\theta}\right) + \frac{\sqrt{D}}{2\sqrt{\pi}\mu E\sqrt{\theta}}\left\{1 + \frac{1}{2}\exp\left[-\frac{(l-\mu E\theta)^2}{4D\theta}\right]\right\}. \tag{A.8}$$

For the contribution to the current due to diffusion one finds

$$\frac{i_D}{i_0} = \frac{\sqrt{D}}{2\sqrt{\pi}\mu E\sqrt{\theta}}\left\{1 + \frac{1}{2}\exp\left[-\frac{(l-\mu E\theta)^2}{4D\theta}\right]\right\}. \tag{A.9}$$

This calculation, however, is pessimistic both because of the simplifications introduced and because of the formulation of the problem. In reality, the beam of ions spreads in a radial sense which limits the contribution of diffusion to the axial current.

RIASSUNTO

È stata sviluppata una nuova tecnica di misura della mobilità di cariche elettriche in liquidi. Essenzialmente si misura il tempo di volo tra una griglia ed un elettrodo collettore connesso ad un elettrometro tramite un opportuno filtro. Questo metodo è di facile applicazione ed i risultati sperimentali sono in eccellente accordo con quelli ottenuti in elio liquido da altri autori.

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Un calcolo sull'assorbimento nucleare di mesoni μ^- in ^{16}O (*).

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(ricevuto il 3 Aprile 1961)

Riassunto. — Usando il «modello a shell» si studia lo spettro d'energia dei neutroni emessi nell'assorbimento nucleare di mesoni μ^- in ^{16}O.

1. – Lo studio dell'assorbimento dei mesoni μ^- da parte di nuclei mantiene un vivo interesse, sia perchè può indicare la forma della interazione non elettromagnetica fra mesoni μ e nucleoni, di gran lunga la meno conosciuta fra le tre classiche interazioni deboli che figurano nel triangolo di Puppi-Wheeler (1) — sia perchè esso può giovare alla conoscenza della struttura nucleare.

Gli stati finali a cui può condurre la reazione, possono, con ovvia notazione, essere così enumerati:

$$
\begin{array}{l}
\text{(I)}\\
\text{(II)}\\
\text{(III)}\\
\text{(IV)}
\end{array}
\quad
(Z, N) + \mu^- \to \left\{
\begin{array}{l}
(Z-1,\, N+1) + \nu\,,\\
(Z-1,\, N) + \mathrm{n} + \nu\,,\\
(Z-2,\, N+1) + \mathrm{p} + \nu\,,\\
(Z-r-1,\, N-s+1) + r\mathrm{p} + s\mathrm{n} + \nu\,, \quad (r+s=2, 3, 4, \ldots)\,.
\end{array}
\right.
$$

(*) Presentato al XLVI Congresso della Società Italiana di Fisica, Napoli, 29 Settembre - 6 Ottobre 1960.

(1) J. TIOMNO and J. A. WHEELER: *Rev. Mod. Phys.*, **21**, 153 (1949).

Le reazioni (III) e (IV) sono molto meno probabili che la reazione (II) poichè richiedono l'intervento di correlazioni dirette fra nucleoni.

Il caso (I) è certamente il più semplice, ma compete favorevolmente con il processo (II) solo per nuclei molto leggeri (2) e, in ogni caso, la possibilità di eseguirne un attendibile studio (3) è fortemente limitata dalla scarsa conoscenza degli stati eccitati del nucleo residuo.

In quest'ordine di idee abbiamo pensato di studiare una reazione del tipo (II) in cui il calcolo esplicito degli elementi di matrice nucleari fosse relativamente sicuro. Abbiamo, per questo, calcolato lo spettro dei neutroni emessi nella reazione

$$^{16}\mathrm{O} + \mu^- \to {}^{15}\mathrm{N} + \mathrm{n} + \nu$$

facendo uso del modello a shell, che sembra particolarmente adatto a descrivere il nucleo in questione (4).

Mentre questo lavoro era in corso ne è stato pubblicato uno alquanto simile, da Dolinsky e Blokhintsev (5). Tralasciamo perciò di esporre il metodo di calcolo da noi usato, rimandando al lavoro suddetto per gli sviluppi formali e ci limitiamo invece a dare conferma di alcuni risultati e a fornire qualche altra indicazione non contenuta nell'articolo in questione.

2. – La scelta da noi fatta per le buche di potenziale relative al ^{16}O e al ^{15}N (Tabella I) differisce sensibilmente da quella di DB.

Per l' ^{16}O la buca di potenziale usata risponde ai seguenti requisiti: *a*) l'energia di legame del livello $1p_{\frac{1}{2}}$ sia uguale alla soglia delle reazioni di estrazione di un protone (12, 11 MeV); *b*) risultino legati i livelli $1d_{\frac{5}{2}}$, $1d_{\frac{3}{2}}$

Tabella I.

	^{16}O	^{15}N
raggio	$4.00 \cdot 10^{-13}$ cm	$4.00 \cdot 10^{-13}$ cm
profondità	34.0 MeV	38.0 MeV

(2) E. G. Beltrametti e L. A. Radicati: *Nuovo Cimento*, **11**, 793 (1959).

(3) A. Fujii e H. Primakoff: *Nuovo Cimento*, **12**, 327 (1959); per una più completa bibliografia si veda anche: H. Primakoff: *Rev. Mod. Phys.*, **31**, 802 (1959).

(4) Quando questo lavoro è stato iniziato, i soli calcoli pubblicati sull'argomento facevano uso del modello a gas di Fermi. Vedi ad es.: H. Überall: *Nuovo Cimento*, **6**, 533 (1957).

(5) E. I. Dolinsky e L. D. Blokhintsev: *Nucl. Phys.*, **10**, 527 (1959) (indicato in seguito con DB).

e $2s_{\frac{1}{2}}$ ([6]). Circa il primo criterio, si suppone che l'interazione di spin-orbita non alteri la funzione d'onda radiale del protone ma soltanto la sua energia e si assume, per la separazione tra i livelli $1p_{\frac{1}{2}}$-$1p_{\frac{3}{2}}$, il valore di 6.3 MeV ([7]).

La scelta della profondità della buca relativa al ^{15}N tiene conto dell'assenza di interazione coulombiana per il neutrone uscente.

Lo spettro dei neutroni emessi è mostrato in Fig. 1 (*). La curva corrispondente alla interazione F-G.T. va confrontata con quella di DB, relativa al valore reale del potenziale sentito dal neutrone uscente. Tenendo conto che i parametri della buca di potenziale usati nei due lavori sono sensibilmente diversi, l'accordo indica che tali calcoli, eseguiti sulla base del modello a shell, posseggono una buona stabilità al variare dei parametri della buca di potenziale. Questa conclusione è anche confermata da un calcolo che abbiamo eseguito con una buca molto diversa da quella prima descritta (raggio $= 4.5 \cdot 10^{-13}$ cm, profondità $= 21$ MeV): a parte irregolarità a basse energie, dovute presumibilmente ad una risonanza, lo spettro ottenuto ha lo stesso andamento per $E_n > 4$ MeV.

Fig. 1. – Spettro d'energia dei neutroni emessi. A V, correzioni incluse $g_P/g_A = +8$; - - - - $A+V$, correzioni incluse, $g_P/g_A = +8$; –·–·– $A - V$, correzioni incluse, $g_P/g_A = -8$; – – – interazione F-G.F.

Una notevole divergenza fra i nostri dati e quelli dei citati autori può riscontrarsi invece nel rapporto tra le probabilità di assorbimento per i protoni degli strati $1s$ e $1p$. Il nostro rapporto è di circa 1 : 90 contro il rapporto 1 : 22 da loro riportato, e la ragione di ciò ci sembra piuttosto oscura.

3. – Accenniamo infine ad alcuni risultati numerici relativi a diverse scelte delle costanti di accoppiamento nella hamiltoniana di interazione. La forma da noi usata per questa hamiltoniana è quella data da PRIMAKOFF ([3]) a cui rimandiamo anche per le notazioni.

([6]) J. P. ELLIOTT e B. H. FLOWERS: *Proc. Roy. Soc.*, A **242**, 57 (1957).
([7]) J. P. ELLIOTT e A. M. LANE: *Handb. d. Phys.*, **39**, 341 (1957).
(*) A causa della elevata energia in gioco, occorre tenere conto anche di momenti angolari del neutrone e del neutrino relativamente elevati. Una rozza valutazione suggerisce tuttavia che si possano trascurare momenti angolari superiori a 3.

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Come mostra la Fig. 1, l'andamento dello spettro è praticamente insensibile all'inclusione o meno delle « correzioni di magnetismo debole » ed al segno relativo delle costanti di accoppiamento assiale e pseudoscalare. I diversi casi differiscono per termini proporzionali a $\nu/2M$ (ν = impulso del neutrino, M = massa del nucleone) nell'ampiezza di transizione. Poichè nell'intervallo d'energia, praticamente accessibile al neutrone, l'impulso del neutrino ha una variazione percentuale solo del 20%, i termini in $\nu/2M$ non producono che lievi modifiche nella forma dello spettro. Non sembra quindi possibile trarre, da osservazioni sperimentali sullo spettro, informazioni sulle costanti di accoppiamento dell'interazione e in particolare sul termine pseudoscalare e su quelli di « magnetismo debole » introdotti da FEYNMAN e GELL-MANN (8) in base all'ipotesi della « corrente vettoriale conservata ».

Contrariamente a quanto accade per la forma dello spettro, la probabilità totale della reazione è piuttosto sensibile alla scelta delle costanti di accoppiamento.

Assumendo, come generalmente accettato, $g_V = 2.7 \cdot 10^{-49}$ erg·cm^3 e $g_A/g_V = -1.25$, le maggiori incertezze riguardano il segno del rapporto g_P/g_A (per cui è stato proposto il valore assoluto 8 (9)) e la presenza o meno delle correzioni di « magnetismo debole ». Può essere perciò interessante dare il valore della vita media per i casi riportati nella Tabella II.

TABELLA II.

Vita media (10^{-5} s)	$A \pm V$	correzioni di « magnetismo debole »	g_P/g_A
1.79	—	incluse	+8
2.04	—	omesse	+8
1.27	—	incluse	—8
1.39	—	omesse	—8
2.70	+	incluse	+8

* * *

Desideriamo esprimere la nostra più viva gratitudine al professore L. A. RADICATI per il costante interesse ed aiuto prestatoci durante questo lavoro.

Siamo grati anche al dr. L. REBOLIA che ha eseguito i calcoli numerici con il calcolatore IBM 650 dell'Università di Bologna.

(8) R. P. FEYNMAN e M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958).

(9) M. L. GOLDBERGER e S. B. TREIMAN: *Phys. Rev.*, **111**, 355 (1958); L. WOLFENSTEIN: *Nuovo Cimento*, **8**, 882 (1958).

SUMMARY

A shell model analysis is given for the energy spectrum of the neutrons ejected in the μ^--absorption by ^{16}O.

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Optical Model Potential for K⁻ Meson-Nucleus Scattering.

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(ricevuto il 4 Aprile 1961)

Summary. — The two-body scattering lengths, which describe the interaction between K^--mesons and free nucleons are used to calculate an optical model potential representing the interaction between K^--mesons and nuclei. The cross-sections for the scattering of K^--mesons by emulsion nuclei are calculated numerically from this potential and are found to conform closely with the experimental values.

1. – Introduction.

Dalitz and Tuan (1) have analysed the K^--proton data, at a laboratory momentum of 175 MeV/c, and have shown that they can be reproduced, for momenta less than 300 MeV/c, equally well by each of four sets of energy independent complex scattering lengths, called the $(a\pm)$ and $(b\pm)$ solutions. Data are available concerning the scattering of K^--mesons by nuclei in emulsions (2-4) and so in order to distinguish between these solutions the K^--meson nuclear cross-sections have been calculated on the basis of the «direct interaction model» (5) using in turn each of these four sets of K^- meson-nucleon scattering lengths.

(1) R. H. Dalitz and S. F. Tuan: *Ann. Phys.*, **10**, 307 (1960).
(2) R. D. Hill, J. H. Hetherington and D. G. Ravenhall: *Phys. Rev.*, **122**, 267 (1961).
(3) M. Melkanoff, D. J. Prowse and D. H. Stork: *Phys. Rev. Lett.*, **4**, 183 (1960).
(4) P. B. Jones: *Proc. Roy. Soc.*, **257**, 109 (1960).
(5) K. M. Watson: *Rev. Mod. Phys.*, **30**, 565 (1958).

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2. - Calculation of the K⁻-meson nuclear optical model potential.

The optical model potential for high energy interactions (as defined by FRANCIS and WATSON [6]) can be approximated in co-ordinate space by

$$V_{\text{op}}(r)=\frac{-2\pi\hbar^2}{m}\left[\frac{mc^2}{E_{\text{K}}\,(\text{lab})}\right]\left[\frac{E_{\text{K}\mathcal{N}}\,(\text{cm})}{Mc^2}\right](1+\Delta)\bar{f}_{\text{cm}}(0)\,\varrho(r)\,, \tag{1}$$

where m, M are the K⁻-meson and nucleon rest masses respectively, $\varrho(r)$ is the nucleon density in the nucleus, $E_{\text{K}}(\text{lab})$ is the total energy of the K⁻-meson in the laboratory frame and $E_{\text{K}\mathcal{N}}(\text{cm})$ is the total energy of the K⁻-nucleon system in their centre of mass frame. $\bar{f}_{\text{cm}}(0)$ is the average forward scattering amplitude in the K⁻-nucleon c. of m. frame, of the nucleons in the nucleus. Δ is a correction factor depending on the position correlations of the nucleons in the nucleus [6]. Eq. (1) includes the relativistic correction factors arising from high energy interactions between the K⁻-meson and the nucleus. The validity of this equation has been discussed by FRANK *et al.* [7].

The average forward scattering amplitude is approximately given by

$$\bar{f}_{\text{cm}}(0)=\frac{Z}{A}\frac{1}{2}(f_0+f_1)+\frac{A-Z}{A}f_1\,, \tag{2}$$

where $f_{0,1}$ are the forward scattering amplitudes for interactions between K⁻-mesons and nucleons for states of isotopic spin $T=0,1$. In eq. (2) the amplitudes are those for scattering from bound nucleons, but at the energies considered it is a good approximation to neglect nuclear binding effects and to use the amplitudes $f_T^{(\text{free})}$ for scattering from free nucleons. The modification to $\bar{f}_{\text{cm}}(0)$ due to the motion of the nucleons in the nucleus is calculated assuming the nucleons have a Fermi momentum distribution.

The K⁻-free proton scattering data are described by energy independent complex scattering lengths A_0, A_1 for $T=0, 1$ states. The corresponding complex s-wave phase shifts δ_T are given by

$$k\,\text{ctg}\,\delta_T=\frac{1}{A_T}+\frac{1}{2}R_Tk^2+\dots\,. \tag{3}$$

Assuming the effective range of the K⁻-nucleon interaction, R_T, is of the order of $\hbar/mc$ the second term may be neglected for energies below 100 MeV. The K⁻-proton data are consistent with pure S-wave scattering at a momentum of

(6) N. C. FRANCIS and K. M. WATSON: *Phys. Rev.*, **92**, 291 (1953).
(7) R. M. FRANK, J. L. GAMMEL and K. M. WATSON: *Phys. Rev.*, **101**, 891 (1956).

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300 MeV/c, but there is evidence of p-wave contributions at 400 MeV/c ([8]). Thus for energies up to 100 MeV $f_T^{(\text{free})}$ may be calculated from

$$(4)\qquad f_T^{(\text{free})} = \frac{A_T}{1 - ikA_T},$$

where k is the K⁻-nucleon centre of mass momentum inside the nucleus.

The first approximation to Δ is ([9]), in units of $c = \hbar = 1$

$$(5)\qquad \Delta = -\frac{iE_K\,(\text{lab})}{p}\, V_{\text{op}}(r) \int_0^\infty G(x)\,\mathrm{d}x\,,$$

where p is the laboratory K⁻ momentum inside the nucleus, and $G(x)$ is the nuclear pair correlation function defined by

$$P_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = P_1(\boldsymbol{x}_1)P_1(\boldsymbol{x}_2)[1 + G(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)]\,,$$

where $P_1(\boldsymbol{x}_1)$ is the probability of finding nucleon «1» at $\boldsymbol{x}_1$ and $P_2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the joint probability of finding nucleon «1» at $\boldsymbol{x}_1$ and nucleon «2» at $\boldsymbol{x}_2$. WATSON and ZEMACH ([9]) estimate $R_c \equiv \int_0^\infty G(x)\,\mathrm{d}x \simeq -\frac{1}{3}r_0$ for a degenerate Fermi gas model of the nucleus of radius $r_0 A^{\frac{1}{3}}$. For a K⁻-meson of energy 80 MeV a typical value of Δ at the centre of the nucleus is $(\Delta)_{r=0} \sim 0.13 \mp 0.10\,i$, $\mp$ according to whether the ($\pm$) solutions of the K⁻-proton data are used. The magnitude of Δ decreases as the energy of the incident meson is increased (see eq. (5)).

In order to ensure that the scattering of the K⁻-meson inside the nucleus takes place only on the energy shell the total mean free path λ_t for either absorption or inelastic collisions of the K⁻-meson must be much greater than the particle wavelength $\lambda\!\!\!^{-}$ inside the nucleus.

Eq. (1) is used to calculate the real part of the optical model potential, but the imaginary part obtained from (1) will be incorrect as the contributions from those collisions prohibited by the nucleons obeying the Pauli exclusion principle will be included. The imaginary part of the potential is correctly obtained from

$$(6)\qquad \operatorname{Im}[V_{\text{op}}(r)] = \frac{-\hbar v}{2\lambda_t}\,,$$

([8]) P. NORDIN: *Thesis*, University of California (1960).
([9]) K. M. WATSON and C. ZEMACH: *Nuovo Cimento*, **10**, 452 (1958).

where v is the K^--meson laboratory velocity inside the nucleus and λ_t, the total mean free path for an inelastic or absorptive collision, is given by

$$(7)\qquad \frac{1}{\lambda_t}=\left[\frac{Z}{A}\left(\eta_p\sigma^p_{el}+\sigma^p_{abs}+\eta_n\sigma_{c.e.}\right)+\frac{A-Z}{A}\left(\eta_n\sigma^n_{el}+\sigma^n_{abs}\right)\right]\varrho(r)\,,$$

where η_p and η_n are the Sternheimer factors [10] for protons and neutrons respectively which account for the reduction of $|\mathrm{Im}[V_{op}(r)]|$ arising from the exclusion principle. The K^--nucleon elastic cross-sections are given by

$$\sigma^p_{el}=\pi|f_0+f_1|^2\,,\qquad \sigma^n_{el}=4\pi|f_1|^2$$

the cross-section for the reaction $K^-+p\to\overline{K}^0+n$ is

$$\sigma_{c.e.}=\pi|f_1-f_0|^2$$

and the K^--nucleon absorption cross-sections are calculated from

$$\sigma^p_{abs}=\tfrac{1}{2}\left[(\sigma_{abs})_{T=0}+(\sigma_{abs})_{T=1}\right]\,,$$

$$\sigma^n_{abs}=(\sigma_{abs})_{T=1}\,,$$

where

$$(\sigma_{abs})_T=\frac{4\pi}{k}\left(\frac{b_T}{1+2kb_T+k^2(a_T^2+b_T^2)}\right).\qquad \text{with } A_T=a_T+ib_T.$$

The real and imaginary parts of the optical model potential are calculated from eq. (1) and (6) respectively. Many factors in these equations depend

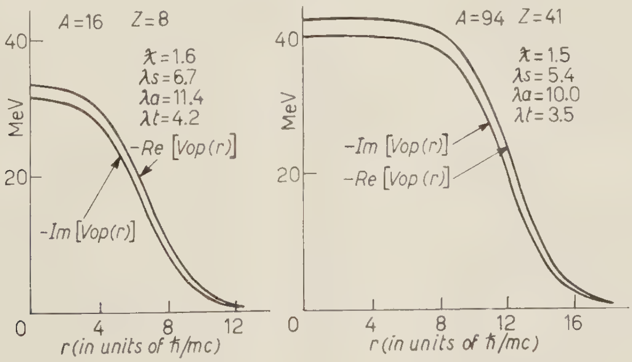

Fig. 1. – The optical model potential for the representative emulsion nuclei obtained using the $(b+)$ K^--proton solution for a 52.5 MeV incident K^--meson, taking $r_0=1.07$ fermi and $a=0.57$ fermi. The values quoted for $\lambda\!\!\!^{-}$, λ_s, λ_a and λ_t are those at the centre of the nucleus, $r=0$, in units of $\hbar/mc$.

([10]) R. M. STERNHEIMER: *Phys. Rev.*, **106**, 1027 (1957).

upon the K⁻-meson energy inside the nucleus, which itself in turn depends on the optical model potential at this point. Consequently it is necessary to compute the real and imaginary parts of the potential by an iterative procedure at each value of r required for the integration of eq. (9). Thus the potential shape $V_{\text{op}}(r)$ will not necessarily be of the same form as the shape of the nucleon distribution $\varrho(r)$. Figs. 1 and 2 show typical optical model poten-

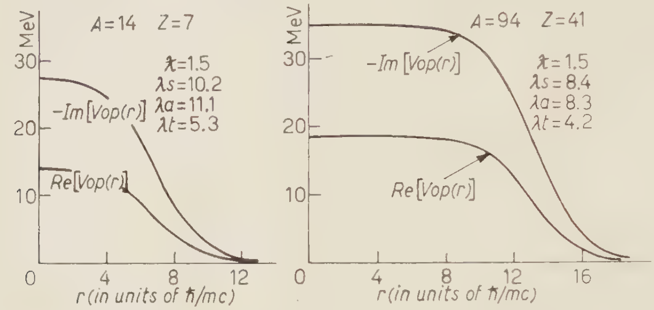

Fig. 2. – The same as Fig. 1, but obtained using the (b —) solution for a 110 MeV incident K⁻-meson, taking $r_0 = 1.18$ fermi and $a = 0.57$ fermi.

tials for the representative « light » and « heavy » nuclei of the emulsion obtained using the ($b+$) and (b —) K⁻-proton solutions respectively. λ_s, λ_a are the mean free paths for inelastic scattering and absorption of the K⁻-meson by the nucleus.

3. – Calculation of the K⁻-meson nuclear cross-sections.

The nucleons in a nucleus (A, Z) of radius $R = r_0 A^{\frac{1}{3}}$ are assumed to have a Woods-Saxon-shaped density distribution

$$\varrho(r) = \frac{\varrho_0}{1 + \exp[(r - R)/a]}, \tag{8}$$

where the parameter a is a measure of the diffuseness of the surface of the distribution. Using this $\varrho(r)$ the potential $V_{\text{op}}(r)$ is calculated as described above throughout the range of r for which $V_{\text{op}}(r)$ is not negligible.

The Klein-Gordon equation for the l-th partial radial wave function $F_l(r)$ is

$$\frac{\mathrm{d}^2 F_l(r)}{\mathrm{d}r^2} + \left[\frac{\{E - V_{\text{op}}(r) - eC(r)\}^2 - (mc^2)^2}{c^2\hbar^2} - \frac{l(l+1)}{r^2}\right] F_l(r) = 0\,, \tag{9}$$

where $C(r)$ is the electrostatic potential between the K⁻-meson and the Saxon-shaped proton distribution and E is the total energy of the incident K⁻-meson.

This equation was solved numerically for all l-values for which a non-negligible complex phase shift δ_l is produced by the complex potential $V_{op}(r)$ and the potential arising from the departure of $C(r)$ from $-Ze/r$.

The elastic differential cross-section $d\sigma/d\Omega$ and the absorption cross-section σ_{abs} for the K$^-$-nuclear scattering are calculated from

$$\frac{d\sigma}{d\Omega} = \left| f_c(\theta) + \frac{1}{\varkappa}\sum_{l=0}^{\infty}(2l+1)\exp\left[i(2\sigma_l+\delta_l)\right]\sin\delta_l P_l(\cos\theta)\right|^2,$$

$$\sigma_{abs} = \frac{\pi}{\varkappa^2}\sum_{l=0}^{\infty}(2l+1)\left[1-\exp\left[-4\,\mathrm{Im}\,(\delta_l)\right]\right],$$

where $\varkappa$ is the K$^-$-nuclear centre-of-mass momentum, σ_l is the Coulomb phase shift and $f_c(\theta)$ is the relativistic Coulomb scattering amplitude for a point charge Ze.

4. – Comparison with experiment.

DALITZ and TUAN [1] have analysed the K$^-$-proton scattering data and find that, within the zero effective range approximation, the data can be reproduced, for momenta less than 300 MeV/c equally well by the four solutions

	A_0 (fermi)	A_1 (fermi)
$(a+)$	$0.2+0.8i$	$1.6+0.4i$
$(a-)$	$-0.3+1.6i$	$-1.0+0.18i$
$(b+)$	$1.6+1.6i$	$0.7+0.22i$
$(b-)$	$-1.8+0.6i$	$-0.33+0.5i$

DALITZ [11] has recently re-analysed the K$^-$-proton data and obtained for the four solutions the following modified complex scattering lengths.

	A_0 (fermi)	A_1 (fermi)
$(a+')$	$0.05+1.1i$	$1.45+0.35i$
$(a-')$	$-0.75+2.0i$	$-0.85+0.21i$
$(b+')$	$1.15+2.0i$	$0.70+0.25i$
$(b-')$	$-1.85+1.1i$	$-0.1+0.65i$

Data concerning the scattering of K$^-$-mesons by emulsion nuclei are available at mean K-meson energies of 52.5 MeV [2], 110 MeV [3] and 125 MeV [4].

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[11] R. H. DALITZ: *Rev. Mod. Phys.* (to be published 1961).

The average emulsion nucleus elastic differential cross-section $d\sigma/d\Omega$ and reaction cross-section σ_{abs} are obtained at these energies by calculating the cross-sections for two representative nuclei and averaging in the nuclear proportions mentioned below. The emulsion used to obtain the data of reference [2] is well represented by two kinds of nuclei, a « light » nucleus $A = 16$, $Z = 8$ and a « heavy » nucleus $A = 94$, $Z = 41$ in the nuclear proportions 0.817 : 0.183. The emulsions of references [3] and [4] are represented by $A = 14$, $Z = 7$ and $A = 94$ $Z = 41$ nuclei in the proportions 0.574 : 0.426 and 0.565 : 0.435 respectively Figs. 3, 4 and 5 show plots of the experimental and calculated values of

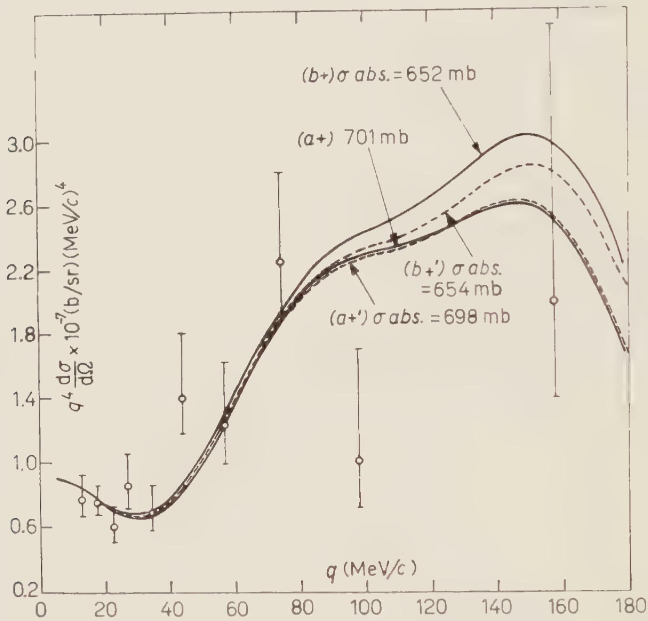

Fig. 3. – The experimental and calculated values of $q^4\, d\sigma/d\Omega$ versus q. The calculated curves were obtained for a 52.5 MeV incident K$^-$-meson using $r_0 = 1.07$ fermi, $a = 0.57$ fermi, $R_c = -\frac{1}{3} r_0$ and the K$^-$-proton solutions indicated. Experimental values obtained by HILL *et al.* (·); experimental $\sigma_{abs} = 640$ mb.

$q^4\, d\sigma/d\Omega$ *vs.* q, the momentum transfer to the nucleus. In Fig. 3 the calculated values of $q^4\, d\sigma/d\Omega$ are obtained for a K-meson of incident energy of 52.5 MeV using only the (+) solutions of the K$^-$-proton data. For the (—) solutions of the K$^-$-proton data the energy of the K$^-$-meson inside the nucleus is considerably less than 50 MeV, consequently the optical model potential cannot be determined using the direct interaction model as described in Section 2. The calculated values of $q^4\, d\sigma/d\Omega$ in Figs. 4 and 5 are obtained using incident K$^-$-meson energies of 110 MeV and 125 MeV respectively. If the (—)

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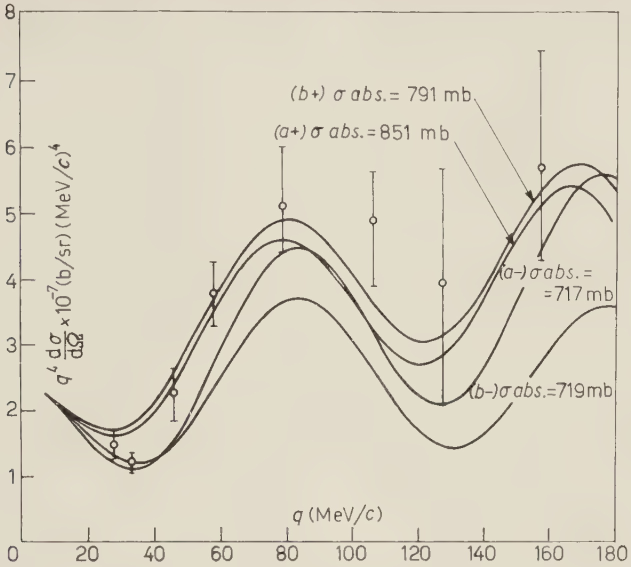

Fig. 4. – The experimental and calculated values of $q^4\,\mathrm{d}\sigma/\mathrm{d}\Omega$ versus q at a mean incident K^--meson energy of 110 MeV. The calculated curves were obtained for $r_0 = 1.18$ fermi, $a = 0.57$ fermi and $R_c = -\frac{1}{3}r_0$. Experimental values obtained by MELKANOFF *et al.* (3); experimental $\sigma_{\text{abs}} = 770$ mb.

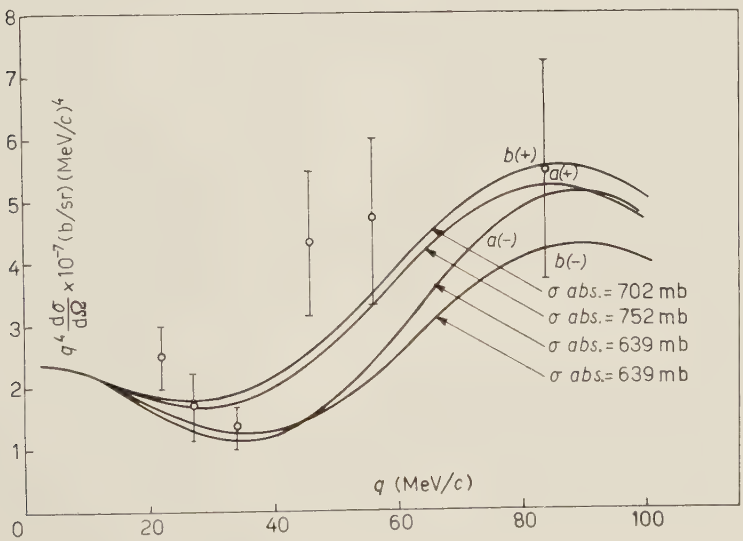

Fig. 5. – The experimental and calculated values of $q^4\,\mathrm{d}\sigma/\mathrm{d}\Omega$ versus q at a mean incident K^--meson energy of 125 MeV. The calculated curves were obtained using $r_0 = 1.07$ fermi, $a = 0.57$ fermi and $R_c = -\frac{1}{3}r_0$. Experimental values obtained by JONES (4); experimental $\sigma_{\text{abs}} = (710 \pm 44)$ mb.

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K$^-$-proton solutions are used for these incident energies, then the energy of the K$^-$-meson at the centre of the nucleus will be about 90 MeV, which is a favourable energy for the model used; but when the (+) K$^-$-proton solutions are taken the meson energy will increase as it enters the nucleus and thus some error may result from a neglect of the K$^-$-nucleon p-wave contribution and use of the zero range K$^-$-nucleon scattering lengths.

A comparison between the calculated results from the K$^-$-proton ($\pm$) and ($\pm'$) solutions are shown in Table I for a K-meson of 110 MeV using $r_0 = 1.18$ fermi and $a = 0.57$ fermi.

TABLE I.

K$^-$-proton solution	« Light » nucleus		« Heavy » nucleus		Calculated σ_{abs} mb
	Re $[V_{op}(0)]$	Im $[V_{op}(0)]$	Re $[V_{op}(0)]$	Im $[V_{op}(0)]$	
$a+$	-17.2	-33.8	-22.4	-43.1	851
$a-$	$+28.0$	-29.4	$+42.6$	-35.0	717
$b+$	-19.6	-25.4	-25.3	-31.4	791
$b-$	$+14.1$	-27.4	$+18.4$	-34.9	719
$a+'$	-17.2	-33.6	-22.6	-42.7	849
$a-'$	$+26.3$	-28.2	$+38.9$	-34.1	707
$b+'$	-18.2	-25.8	-23.7	-32.0	791
$b-'$	$+6.3$	-29.2	$+7.7$	-37.2	752

$V_{op}(0)$ is the optical model potential at the centre of the nucleus in MeV. The calculated σ_{abs} is to be compared with the experimental value of 770 mb.

Fig. 6 shows the variation of the emulsion cross-section, calculated using the ($b+$) solution, produced by altering R_c and the parameters r_0 and a of eq. (8). It is seen that the calculated values of $d\sigma/d\Omega$ and σ_{abs} depend on the choice of these nuclear parameters and thus when more K$^-$-nucleon and K$^-$-nucleus data are available the purpose of this calculation may be altered to yield information on the size of these parameters.

The comparison of the calculated cross-sections with experiment in Figs. 3, 4 and 5 seem to favour the ($b+$) solution to the K$^-$-proton data, but because of the uncertainty in the experimental values of the nuclear emulsion differ-

DOMUS GALILÆANA

PISA - VIA S. MARIA, 18

CONCORSO INTERNAZIONALE

AL

"PREMIO GALILEO, 1964"

DI QUATTRO MILIONI DI LIRE ITALIANE

voltare

DOMUS GALILÆANA

PISA - VIA S. MARIA, 18

CONCORSO INTERNAZIONALE

AL

"PREMIO GALILEO, 1964"

DI QUATTRO MILIONI DI LIRE ITALIANE

La Domus Galilæana, prendendo occasione della ricorrenza, nel 1964, del IV Centenario della nascita di Galileo Galilei, col fine di onorare la memoria di lui e favorire gli studi galileiani, mette in palio, con Concorso Internazionale, un premio di 4 000 000 (quattro milioni) di lire italiane, netti e indivisibili, intitolato « Premio Galileo, 1964 », da assegnarsi all'Autore o agli Autori di quella monografia che, fra tutte quelle presentate al Concorso secondo le norme qui sotto indicate ai n. 1 e 2, sia, da parte della Commissione Giudicatrice, non solo dichiarata la migliore allo *scopo precipuo* di *Far rivivere e illustrare sotto l'aspetto storico-critico, in modo ampio, profondo e documentato, l'opera di Galileo Galilei e il suo pensiero scientifico*, ma dichiarata altresì *degna*, in valutazione assoluta, del Premio.

Il Concorso è aperto a chiunque voglia partecipare, qualunque sia la sua nazionalità.

La partecipazione al Concorso implica da parte dei singoli concorrenti l'accettazione incondizionata delle seguenti norme.

1. Ogni monografia presentata al Concorso:

   a) può essere opera di uno o più Autori;

   b) deve essere inedita al momento della presentazione e tale restare almeno fino al termine del Concorso;

   c) deve avere un'estensione di circa 500 pagine dattiloscritte di 30-35 righi ciascuna e 60-65 battute per rigo (ciò che equivale grosso modo a un comune volume a stampa di circa 300 pagine);

   d) deve essere divisa in capitoli e paragrafi forniti di titoli e sottotitoli, e corredata almeno di due indici: uno sistematico di tutta la monografia e uno analitico, abbastanza esteso, delle cose notevoli, ordinato per nomi di persone citate e per argomenti;

   e) può contenere in testo e fuori testo figure, tabelle e appendici;

   f) deve essere corredata, lungo il testo, di note che servano a indicare, in modo preciso e completo, le fonti bibliografiche e documentarie usate;

   g) deve usare di regola, come testo critico degli scritti di Galileo, l'*Edizione Nazionale delle Opere di Galileo Galilei* (Editore Barbèra - Firenze) e ad essa fare i riferimenti;

h) deve essere presentata *in due sole copie*: una o in lingua italiana o in lingua francese e l'altra in una delle seguenti lingue — francese, inglese, italiana, portoghese, russa, spagnola, tedesca — scelta in modo però che non sia quella medesima usata nella prima copia; inoltre le due copie debbono essere scritte a macchina in inchiostro nero su fogli staccati e su una pagina sola di essi;

i) non deve portare indicazione alcuna del nome dell'Autore o degli Autori;

l) deve essere contrassegnata con un motto (scritto con caratteri latini in stampatello) e accompagnata da una busta chiusa che *all'esterno* rechi il motto, senza altra indicazione, e *all'interno* contenga un foglio sul quale sia ripetuto il motto e siano anche indicati (con caratteri latini in stampatello) il nome, il cognome e il recapito dell'Autore (o, se la monografia è opera di più Autori, i nomi, i cognomi e i recapiti degli Autori) e il nome, il cognome e il recapito della persona (o la denominazione e la sede dell'Ente) alla quale (o al rappresentante legale del quale) nel caso che la monografia risulti vincitrice, deve essere materialmente consegnata la somma in palio.

2. Le due copie di cui alla norma 1 - *h*) debbono essere inviate, *possibilmente in un unico pacco*, al seguente indirizzo: « Presidenza Domus Galilæana - Concorso Galileo - Via S. Maria, 18 - Pisa (Italia) », *e giungere a destinazione non prima delle ore 12 di Giovedì 1° Agosto 1963 e non dopo le ore 12 di Lunedì 30 Settembre dello stesso anno.*

3. La Commissione Giudicatrice è composta del Presidente della Domus Galilæana, del Direttore della Domus Galilæana e di altri tre cultori di Storia della Scienza, scelti, dopo il 30 Settembre 1963, dal Consiglio di Amministrazione della Domus.

4. La Commissione Giudicatrice stabilirà dapprima, senza possibilità di ex-æquo, quale delle monografie presentate al Concorso meglio dell'altre corrisponda allo *scopo precipuo* indicato nell'introduzione di questo bando; indi giudicherà se essa, in valutazione assoluta, è *degna* del Premio: in caso affermativo la Commissione la dichiarerà vincitrice del Concorso stesso, e, al fine di conoscerne l'Autore o gli Autori, aprirà la busta (e solo quella) che, secondo la norma 1 - *l*), accompagna la monografia stessa; in caso negativo nessuna busta verrà aperta e il Concorso sarà dichiarato senza esito e potrà essere ripetuto.

5. Il parere della Commissione è inappellabile.

6. Tutte le copie delle monografie inviate e le relative buste saranno conservate nell'archivio della Domus.

7. La proclamazione del vincitore (o, se la monografia vincitrice è opera di più Autori, la proclamazione dei vincitori) avverrà, da parte della Presidenza della Domus, nel 1964, e nello stesso anno avrà anche luogo, durante una cerimonia celebrativa alla Domus, la consegna materiale del Premio alla persona (o al rappresentante dell'Ente) che, nella busta che accompagna la monografia vincitrice, è indicata (o è indicato) a questo scopo: a detta persona (o a detto rappresentante) verranno rimborsate la spesa di viaggio di andata e ritorno dalla sua residenza abituale fino a Pisa e quella di soggiorno a Pisa per i giorni della cerimonia.

8. La Domus si riserva il diritto di pubblicare, a proprie spese e nelle proprie edizioni, la monografia vincitrice, scegliendo a suo piacimento una delle lingue o tutte e due le lingue nelle quali essa è stata presentata. All'Autore o agli Autori della monografia vincitrice, nulla è dovuto per questo diritto della Domus.

Dalla Domus Galilæana
il 4 Luglio 1960

IL PRESIDENTE
G. Polvani

DOMUS GALILÆANA

PISA - VIA S. MARIA, 18.

CONCOURS INTERNATIONAL

"PREMIO GALILEO, 1964"

(« PRIX GALILEE, 1964 »)

DE QUATRE MILLIONS DE LIT.

La Domus Galilæana à l'occasion du quatrième centenaire, en 1964, de la naissance de Galileo Galilei, dans le but d'honorer sa mémoire et d'encourager les études galiléennes, organise un Concours International, doté d'un prix de 4.000.000 de lires (quatre millions de lires), net et indivisible, intitulé « Premio Galileo, 1964 », qui sera décerné à l'Auteur ou aux Auteurs de la monographie qui, parmi toutes celles qui seront présentées au Concours, conformément aux règlements indiqués ci-dessous aux articles 1 et 2, sera déclarée par le Jury non seulement la meilleure quant au *but essentiel* qui sera de *Faire revivre et d'illustrer, sous l'aspect historique et critique, d'une façon ample, approfondie et documentée, l'oeuvre et la pensée scientifique de Galileo Galilei*, mais encore *digne* en valeur absolue, du Prix.

Le Concours est ouvert à quiconque désire y participer, sans condition de nationalité.

La participation au Concours suppose, de la part de chaque concurrent, l'acceptation inconditionelle des règlements suivants.

1. Toute monographie présentée au Concours:

a) peut être l'oeuvre d'un ou plusieurs Auteurs ;

b) doit être inédite au moment de la présentation et doit le rester au moins jusqu'à la fin du présent Concours;

c) doit compter environ 500 pages dactylographiées, de 30-35 lignes chacune, à raison de 60-65 frappes par ligne (ce qui équivaut grosso modo à un volume imprimé courant d'environ 300 pages);

d) doit être divisée en chapitres et paragraphes pourvus de titres et de sous-titres et suivie d'au moins deux index: l'un systématique de toute la monographie, l'autre analytique, assez abondant, des matières importantes, ordonnées alphabétiquement par noms des personnes nommées et par arguments;

e) peut comprendre en texte et hors texte, des illustrations, tableaux et appendices;

f) doit être accompagnée, tout au long du texte, de notes servant à indiquer, de façon précise et complète les sources bibliographiques et la documentation utilisées;

g) doit se servir de préférence, comme texte critique des écrits de Galilée, de l'*Edizione Nazionale delle Opere di Galileo Galilei* (Ed. Barbèra - Firenze) et s'y référer;

h) doit être présentée *en deux exemplaires* seulement: l'un en italien ou en français, l'autre dans une des langues suivantes: allemand, anglais, espagnol, francais, italien, portugais, russe, de façon que la langue choise pour le second exemplaire soit différente de celle du premier; en outre, les deux exemplaires doivent être tapés à la machine avec un ruban noir au recto seulement de feuillets simples;

i) ne doit porter aucune indication du nom de l'Auteur ou des Auteurs;

l) doit être marquée d'une devise (écrite en caractères latins d'imprimerie) et accompagnée d'une enveloppe fermée portant *à l'extérieur* la devise sans autre indication, et contenant *à l'intérieur* une feuille sur laquelle est répétée la devise et seront indiqués (en caractères latins d'imprimerie) le nom, le prenom et l'adresse de l'Auteur (ou des Auteurs) et le nom, le prénom e l'adresse de la personne (ou la dénomination et le siège de l'Organisme) à laquelle (ou au représentant légal duquel), dans le cas où la monographie serait déclarée gagnante, doit être matériellement remis le montant du Prix.

2. Les deux exemplaires, dont il est parlé au règlement 1, *h*, doivent être envoyés, *si possible en un paquet unique*, à l'adresse suivante: « Presidenza Domus Galilæana - Concorso Galilei - Via S. Maria 18, Pisa (Italia) »; *et arriver à destination ni avant 12 heures du Jeudi 1er Août 1963, ni après 12 heures du Lundi 30 Septembre de la même année.*

3. Le Jury est composé du Président de la Domus Galilæana, du Directeur de la Domus Galilæana et de trois autres spécialistes d'Histoire de la Science, choisis aprés le 30 Septembre 1963, par le Conseil d'Administration de la Domus.

4. Le Jury établira d'abord, sans possibilité d'ex-aequo, la monographie qui, parmi celles qui sont présentés au Concours correspond le mieux au *but essentiel* indiqué ci-dessus, puis il jugera si celle-ci, considerée quant à sa valeur absolue, est *digne* du Prix. Dans l'affirmative, le Jury la déclarera gagnante et, afin d'en connaître l'Auteur ou les Auteurs, il procédera à l'ouverture de l'enveloppe (et de celle-là uniquement) qui accompagne, conformément au réglement 1 - *l*, la monopraphie elle-même; dans le cas contraire, aucune enveloppe ne sera ouverte, le Concours sera déclaré nul et pourra ainsi être répété.

5. La décision du Jury est sans appel.

6. Tous les exemplaires des monographies envoyées au Concours et les enveloppes fermées qui leur sont relatives seront conservés dans les archives de la Domus.

7. La proclamation du gagnant (ou, si la monographie primée est l'oeuvre de plusieurs Auteurs, la proclamation des gagnants) sera faite par la Présidence de la Domus, en 1964, et la même année aura lieu également, au cours d'une cérémonie qui se déroulera à la Domus, la remise effective du Prix à la personne qui, dans l'enveloppe accompagnant la monographie primée, est indiquée à cet effet: à cette personne seront remboursés les frais de voyage aller et retour de sa résidence habituelle jusqu'à Pise ainsi que ses frais de séjour à Pise pour les jours de la cérémonie.

8. La Domus se réserve le droit de publier à ses propres frais et dans ses propres éditions, la monographie primée, en choisissant à son gré une des langues ou les deux langues dans lesquelles elle a été présentée. La Domus ne devra rien à l'Auteur ou aux Auteurs de la monographie pour la jouissance de ce droit.

De la Domus Galilæana
le 4 Julliet 1960.

LE PRESIDENT
G. POLVANI

DOMUS GALILÆANA

PISA - VIA S. MARIA, 18

INTERNATIONAL COMPETITION

FOR THE

"PREMIO GALILEO, 1964"

(« GALILEO PRIZE, 1964 »)

OF FOUR MILLION LIRE.

On the occasion of the fourth centenary, in 1964, of the birth of Galileo Galilei, the Domus Galilæana, in his memory and to encourage Galilean studies, announces an International Competition for a net and indivisible prize of Lit. 4.000.000 (four millions) entitled « Premio Galileo, 1964 ». It is to be awarded to the Author or Authors of the monograph which, of all those presented at the Competition in accordance with rules 1 and 2 stated below, is deemed by the judging Committee to be best, not only from the point of view of the primary purpose of *Illustrating and reviving interest in the work and scientific thought of Galileo Galilei with an ample, deep, and documented treatment of a historical and critical type*, but also from the point of view of *general excellence and absolute merit.*

The Competition is open to everyone of any nationality who wishes to participate.

Participation in the competition implies on the part of each competitor unconditional acceptance of the following rules.

1. Every monograph presented to the Competition:

   a) may be the work of one or more Authors;

   b) shall be unpublished at the moment of presentation and until the end of the present Competition;

   c) shall have a length of about 500 pages of 30-35 lines each and 60-65 strokes per line (which corresponds to a common printed volume of about 300 pages);

   d) shall be divided in chapters and paragraphs with titles and subtitles, and furnished with at least two indexes: a systematic one of the whole monograph and a fairly extended analytic index including, in alphabetic order, notable things, topics, and names of persons cited;

   e) may contain, in text and out of text, figures, tables and appendices;

   f) shall include, along with the text, footnotes which serve to indicate exactly and completely the bibliographic and documentary sources used;

   g) shall, as a rule, refer to the *Edizione Nazionale delle Opere di Galileo Galilei* (Publisher Barbèra - Firenze) as the critical text of Galileo's writings and refer to it in the footnotes;

h) shall be presented in *only two copies*: one in Italian or French and the other in an additional language which is one of the following group: English, French, German, Italian, Portuguese, Russian, Spanish; the two copies shall be typewritten with black ink on only one side of loose sheets;

i) shall be marked with a motto (written in latin block letters) and accompanied by a closed cover which *on the outside* bears only the same motto and *on the inside* contains a sheet on which the motto is repeated and where may be found (written in latin block letters) the name and the address of the Author (or if the monograph is the work of more than one Author, the names and the addresses of the Authors) and the name and the address of the person (or of the Institution) to whom (or to the legal representative of which) the amount of the Prize should be paid in case the monograph is declared winner of the Prize.

2. The two copies as per rule 1-*h* shall be sent, *if possible in only one parcel*, to the following address: « Presidenza Domus Galilæana - Concorso Galilei - Via S. Maria, 18 - Pisa (Italia) »; *and reach their destination not before 12 o'clock of Thursday, August 1st, 1963 and not after 12 o'clock of Monday September 30th of the same year.*

3. The judging Committee is formed by the President of the Domus Galilæana, the Director of the Domus Galilæana and three other experts in History of Science, chosen, after September 30th, 1963 by the Domus' Board of directors

4. The judging Committee will first decide which of the monographs presented best fulfils the *primary purpose* stated in the introduction of this announcement, the occurrence of a tie being excluded. Then it will decide if the monograph so chosen *deserves* the Prize on the basis of general excellence and absolute merit. If the latter decision is in the affirmative the monograph will be declared winner of the Competition and, to determine its Author or Authors, the Committee will open the cover (and only that one) which, after rule 1-*l*) accompanies the monograph; if the decision is in the negative no cover will be opened and the Competition will be declared issueless, and may be repeated.

5. The decision of the Committee is final.

6. All the copies of the monographs presented to the Competition and the accompanying covers will be filed in the archives of the Domus.

7. The proclamation of the winner (or, if the winning monograph is the work of more than one Author, of the winners) will be made by the President of the Domus in 1964. In the same year, during a ceremony of celebration arranged by the Domus Galilæana, the Prize will be delivered to the person who, in the cover which accompanies the winning monograph, is designated for that purpose. This person will be reimbursed for travelling expenses from his customary residence to Pisa and return, and for his stay at Pisa for the days of the ceremony.

8. The Domus reserves the right to pubish, at its own expense and in its own editions, the winning monograph, choosing either one or both of the languages in which it has been presented. The Author or Authors will be due no compensation for these rights of pubblication reserved by the Domus.

From the Domus Galilæana
given July 4th, 1960

THE PRESIDENT
(G. Polvani)

DOMUS GALILÆANA

PISA - VIA S. MARIA, 18

CONCORSO INTERNAZIONALE

AL

"PREMIO GALILEO, 1964"

DI QUATTRO MILIONI DI LIRE ITALIANE

voltare

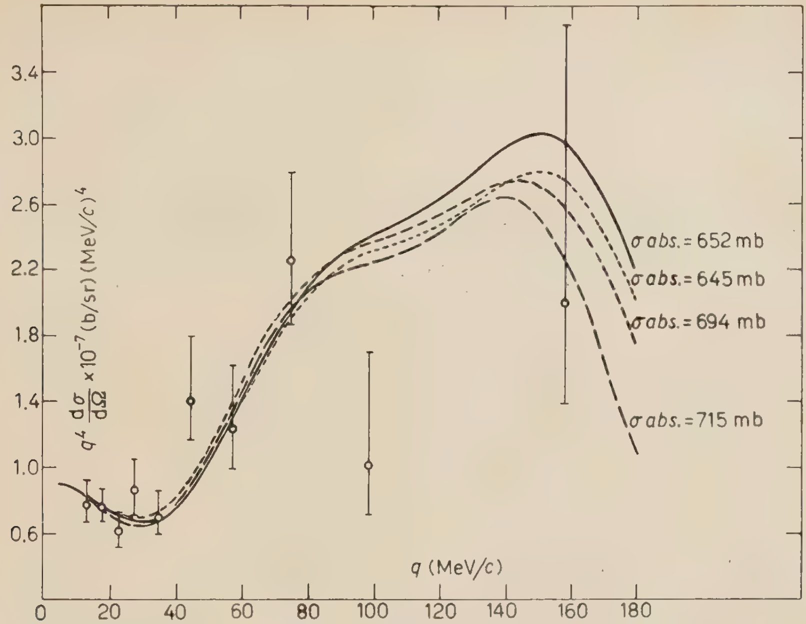

Fig. 6. – The curves are the values of $q^4\,\mathrm{d}\sigma/\mathrm{d}\Omega$ versus q calculated for different values of the nuclear parameters using the $(b+)$ solution for a 52.5 MeV incident meson. ϕ experimental results of HILL *et al.* (·): —— $r_0=1.07$, $a=0.57$ fermi, $R_j=-\frac{1}{3}r_0$; — — $r_0=1.18$, $a=0.57$ fermi, $R_j=-\frac{1}{3}r_0$; - - - $r_0=1.07$, $a=0.68$ fermi, $R_j=-\frac{1}{3}r_0$; - · - · - $r_0=1.07$, $a=0.57$ fermi, $R_j=0$.

ential cross-section and in the size of the nuclear parameters r_0 and a this result is not conclusive.

* * *

The writer wishes to thank Professor E. H. S. BURHOP, who suggested the calculation, Professor Sir HARRIE MASSEY and Professor R. D. HILL for their help and guidance in the research. The author is also indebted to the University of London Computer Unit for making time available on their Ferranti Mercury computer, and to the University of London for a studentship.

RIASSUNTO (*)

Per calcolare un potenziale di modello ottico, che rappresenti l'interazione fra mesoni K⁻ e nuclei, uso le lunghezze di scattering di due corpi, che descrivono l'interazione fra mesoni K⁻ e nucleoni liberi. Calcolo numericamente in base a questo potenziale la sezione d'urto per lo scattering dei mesoni K⁻ sui nuclei dell'emulsione e trovo che si accordano molto bene con i valori sperimentali.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO VOL. XXI, N. 1 1° Luglio 1961

Remarks on the Nuclear Absorption of Negative Pions in Deuterium.

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(ricevuto il 10 Aprile 1961)

Summary. — The correlation between the polarization of the neutrons emitted in the capture reaction $\pi^- + d \to 2n$ is examined. It is found to be simply related to the state of the absorbed pion, providing a means of separating the contributions to the process from s and p states of the meson. A possible extension to the capture in complex nuclei is briefly sketched.

1. – The nuclear absorption of negative pions, captured by the Coulomb field in orbits very close to the nucleus, is among the most direct sources of information about the pion-nucleon interaction at very low energies.

Unfortunately, the partial ignorance of the nuclear structure and of the orbital state of the absorbed pion introduces strong limitations into the penomenological study of the interaction.

This paper is intended to discuss a possibility of distinguishing the absorption processes relative to the pion in an s or in a p Coulomb orbit (*).

A similar purpose has already been taken into consideration, in respect to complex nuclei in which the meson's optical transitions give rise to X-rays of rather high energy. The idea is to measure the width of the K- and L-lines in order to deduce the pion mean-life in the different shells. This is near the limit of experimental possibility and has not yet been able to provide definite conclusions (1).

(*) The shortened forms of « s-absorption » and « p-absorption » will be used.
(1) J. M. CASSELS: *Suppl. Nuovo Cimento*, **14**, 259 (1959).

Let us remember that the mechanism, proposed by BRUECKNER *et al.* [2,3] to describe the non-radiative absorption of the π-meson, requires a direct collision of two nucleons in order to absorb the large energy and the small momentum available. The validity of such a model has been reconfirmed recently in an experiment by OZAKI *et al.* [4]. They found that, in complex nuclei, the pion absorption probability by a proton-neutron pair (a « quasi-deuteron ») is about five times larger than by a proton-proton pair; moreover, the two nucleons emerge from the nucleus in opposite directions, within the experimental errors.

Now, if we consider the case of pion absorption by a quasi-deuteron, we could expect that the antisymmetry requirement for the final state of the two ejected neutrons, together with angular momentum and parity conservation, causes the spin-state of the emitted neutrons to be different, according to whether the pion is absorbed in an s or a p orbit.

Of course, these predictions will suffer from the poor knowledge of the nuclear wave functions, expecially of the two-nucleon correlation function. Things appear more simple for the pion absorption in deuterium, since in this case the problem of the detailed nuclear structure is missing.

Therefore we shall examine (in Sections **2** and **3**), the possibility of obtaining information about the non-radiative s- and p-absorption in deuterium, by means of correlation measurements in the polarization of the final neutrons.

This purpose might appear useless, bearing in mind that some calculations on the importance of the p-absorption in deuterium can be found in the literature of about ten years ago. This occurred in connection with some experiments [5] which gave the value ~ 2.36 for the ratio between the probabilities of the processes $\pi^- + d \to 2n$ and $\pi^- + d \to 2n + \gamma$ (*). An s-absorption contribution to the non-radiative process would imply a negative parity of the (charged) pion, relative to the nucleon. This was the main reason to investigate if the s-absorption would play a role or not. It is well known that the answer was yes, after BRUECKNER, SERBER and WATSON [2] estimated that

(2) K. A. BRUECKNER, R. SERBER and K. M. WATSON: *Phys. Rev.*, **81**, 575 (1951); R. E. MARSHAK: *Rev. Mod. Phys.*, **23**, 137 (1951).

(3) K. A. BRUECKNER, R. SERBER and K. M. WATSON: *Phys. Rev.*, **84**, 258 (1951); K. A. BRUECKNER, R. J. EDEN and N. C. FRANCIS: *Phys. Rev.*, **98**, 1455 (1955).

(4) S. OZAKI R. WEINSTEIN, G. GLASS, E. LOH, L. NEIMALA and A. WATTENBERG: *Phys. Rev. Lett.*, **4**, 533 (1960).

(5) W. K. H. PANOFSKY, R. L. AAMODT and J. HADLEY: *Phys. Rev.*, **81**, 565 (1951); W. CHINOWSKY and J. STEINBERGER: *Phys. Rev.*, **95**, 1561 (1954); J. A. KUEHNER, A. W. MERRISON and S. TORNABENE: *Proc. Phys. Soc.*, **73**, 551 (1958).

(*) The ratio of the non-radiative to the radiative capture probability increases very rapidly with the nuclear mass number, *e.g.* for the capture in carbon, its value is ~ 65. It should also be noted that the process $\pi^- + d \to 2n + \pi^0$ is unimportant [6].

(6) W. CHINOWSKY and J. STEINBERGER: *Phys. Rev.*, **100**, 1476 (1955).

the p-absorption probability is smaller, by about one order of magnitude, than the probability of the electromagnetic transition to the s-state.

It must be realized that in the above-mentioned calculation the transition matrix element is assumed to be a linear function of the pion momentum; moreover the distortion of the Coulomb wave-function over the nuclear volume is neglected.

Therefore it does not seem useless to look for direct information on the relative importance of the s- and p-absorption in deuterium. Beside the interest of the problem in itself, the possibility of extending the analysis to complex nuclei (in which the p-absorption is certainly important) is to be taken into account. We shall say few words about this last point in Section **4**.

2. – Here the possible correlations in the polarization of the two neutrons coming from

$$\pi^- + \mathrm{d} \to 2\mathrm{n}$$

will be analysed for the pion absorption in an s or a p orbit.

The two-neutron state, relative to orbital angular momentum L, spin S, total angular momentum J and third component M, will be denoted by

$$y_{LSJ}^{M} = \sum_{m+\mu=M} C_{L\,S\,J}^{m\,\mu\,M}\, Y_{Lm}(\theta, \varphi)\, \chi_{S\mu}\,, \tag{1}$$

where $C_{L\,S\,J}^{m\,\mu\,M}$ are the Clebsch-Gordan coefficients, Y_{Lm} is the spherical harmonic function of order L, m, depending on the relative angular co-ordinates, and $\chi_{S\mu}$ is the spin function relative to the value μ for the third component.

Of course no preferred direction will be supposed for the mesic atom. Therefore the state of the two neutrons will be an incoherent superposition, with equal weight, of the wave functions y_{LSJ}^{M} corresponding to all allowed M-values.

Let α and α', (β and β'), be the spin-up, (spin-down), states relative to the quantization axes z and z' chosen to define the spin of the neutrons. These axes are determined by the arrangement of the analysers A and A' used to measure the polarization of the two neutrons. For the sake of definiteness the index 1 will refer to the neutron detected in A and the index 2 to that one detected in A' (see Fig. 1).

The probability of the situation in which neutron 1 is observed, *e.g.*, in the spin state α and neutron 2 in the state α', is proportional to

$$\sum_{M} |[\alpha(1)\alpha'(2), y_{LSJ}^{M}]|^2\,,$$

where the square bracket signifies integration over the spin co-ordinates. As previously pointed out, the whole state of the two particles is assumed to be

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a mixture of the wave function y_{LSJ}^{M} corresponding to $M=\pm J$, $\pm(J-1)$, ..., ± 1, 0.

It is then natural to define the correlation degree between the neutron polarizations as follows:

$$P_{LSJ}=\frac{\sum_M |[\alpha(1)\alpha'(2),\, y_{LSJ}^{M}]|^2-\sum_M |[\alpha(1)\beta'(2),\, y_{LSJ}^{M}]|^2}{\sum_M |[\alpha(1)\alpha'(2),\, y_{LSJ}^{M}]|^2+\sum_M |[\alpha(1)\beta'(2),\, y_{LSJ}^{M}]|^2}, \tag{2}$$

(the substitution of $\alpha(1)$ by $\beta(1)$ simply introduces a sign change in the definition of P_{ISJ}).

The angular momentum and parity conservation in the reaction

$$\pi^- + \mathrm{d} \to 2\mathrm{n}$$

together with the Pauli principle requirements, do impose severe restrictions on the final state. Indeed, it must be a mixture of the states y_{111}^{M}, $(M=\pm 1, 0)$. if the pion is in an s orbit, while, in the case of p-absorption, it must be a misture of y_{000}^{0} and y_{202}^{M}, $(M=\pm 2\ \pm 1,\ 0)$.

Therefore, denoting P^s and P^p the correlation degrees between the neutron polarizations for s-absorption and p-absorption respectively, we can write

$$P^s = P_{111}, \tag{3}$$

$$P^p = |a|^2 P_{000} + |b|^2 P_{202}, \qquad \left(|a|^2+|b|^2=1\right), \tag{4}$$

where $|a|^2$ and $|b|^2$ are the probabilities for the system (deuteron+pion in a p orbit) to have a total angular momentum $J=0$ of $J=2$.

The relations (3) and (4), together with the definitions (1) and (2), allow a straightforward calculation of P^s and P^p.

For the sake of simplicity, the co-ordinate system adopted to define the states y_{LSJ}^{M} (see eq. (1)) will be chosen to have the quantization axis z coincident with the axis identified by the analyser A, as shown in Fig. 1.

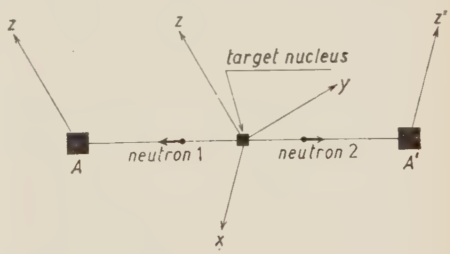

Fig. 1.

In other words, in performing the integrations indicated in eq. (2), the spin functions $\chi_{s\mu}$, α' and β' will be expressed through α and β.

If we are interested in the transverse polarizations of the neutrons, *i.e.* if the axes z and z' are perpendicular to the emission direction of the neutrons,

the spherical harmonic functions in eq. (1) and (2) have to be calculated for $\theta = \pi/2$. Alternatively if we are interested in the case of longitudinal polarization, *i.e.* if the axes z and z' are parallel to the emission direction of the neutrons, the functions Y_{Lm} have to be calculated for $\theta = 0$.

The explicit calculation of P^s, (eq. (3)), gives a vanishing value in the case of transverse polarizations, no matter what the angle between the axis z and z'. Instead, in the case of longitudinal polarizations and taking z' coincident with z, one obtains $P^s = +1$. In other words, if the spin of neutron 1 is parallel, (or anti-parallel), to its momentum, then the spin of neutron 2 is anti-parallel, (or parallel), to its own momentum.

The calculation of P^p is quite obvious and does not depend on the quantities $|a|^2$ and $|b|^2$ which appear in eq. (4). This is because $y_{000}^{\;0}$ and $y_{202}^{\;M}$ are both singlet spin-functions, so that

$$P^p = \frac{|[\alpha(1)\alpha'(2), \chi_{00}]|^2 - |[\alpha(1)\beta'(2), \chi_{00}]|^2}{|[\alpha(1)\alpha'(2), \chi_{00}]|^2 + |[\alpha(1)\beta'(2), \chi_{00}]|^2} = \frac{|[\alpha'(2), \beta(2)]|^2 - |[\beta'(2), \beta(2)]|^2}{|[\alpha'(2), \beta(2)]|^2 + |[\beta'(z), \beta(2)]|^2}.$$

The correlation degree P^p depends only on the relative directions of the axes z and z', but does not depend on the angle between z and the motion direction of the neutrons. This is due to the fact that the singlet function χ_{00} is invariant under rotation of the co-ordinate axes.

The highest correlation is obtained when z and z' are parallel: in fact in this case one obtains $P^p = -1$ if the two axes have the same direction ($P^p = +1$ in the opposite case). In other words, if neutron 1 is observed to have spin-up, then neutron 2 has certainly spin-down, no matter which is the direction of the quantization axis.

From the experimental point of view the measurement of transverse polarizations is the simpler one. Therefore, limiting ourselves to this particular case, we can summarize the situation by saying that there is no correlation between the two-neutron polarizations if the pion is absorbed in an s orbit; instead there is complete correlation in the case of p-absorption.

The experimental possibility, of measuring in this way the relative importance of the s- and p-absorption, obviously depends on the efficiencies η and η' of the analysers A and A' (*).

If we are not wrong, there are not yet tested analysers for neutron polarization in the energy region about 70 MeV (this is actually, in our case, the energy of the neutrons). However, the right-left asymmetry in the elastic scattering in carbon gives a value $\eta \sim 0.3$ for 70 MeV protons (7) and there

(*) Denoting by r the probability for a spin-up neutron to be recognized in such a spin state by the analyser, the efficiency η is defined by $\eta = r - (1 - r) = 2r - 1$.

(7) L. Wolfenstein: *Ann. Rev. Nucl. Sci.*, **6**, 43 (1956); A. E. Taylor: *Rep. Prog. Phys.*, **20**, 125 (1957).

is no reason to believe that things are much different for neutrons. Higher efficiency could perhaps be obtained with neutron scattering in helium, as suggested by the data at energies below ~ 25 MeV ([8]).

In the case of complete correlation, the detectable effect is given by the product of the efficiencies η and η'. Its order of magnitude could be at least 10 %.

3. – In the preceding section we discussed the possibility of distinguishing the s-absorption and the p-absorption of the π^--meson, by means of coincidence measurements of the neutron polarization. Let f_s and f_p be the probabilities of the two processes.

The question is now to know what information can be deduced about the corresponding transition probabilities per unit time T_s and T_p. These are indeed more immediately related to the Hamiltonian interaction.

Up to now we spoke of s and p meson-orbits, without needing to specify the radial quantum number. However it is to be expected that only the deeper orbits do contribute to the absorption process. Therefore, in order to simplify, we shall refer to the 1s and 1p states.

The meson transitions from one orbit to a lower one occur mainly by Auger effect ([9-11]). However, when the radial quantum number is less than ~ 3, the meson orbit becomes smaller than the electron K-orbit and the electromagnetic transition, with X-ray emission, prevails ([9]). Because of the selection rule for the orbital quantum number, $\Delta l = \pm 1$, the s-state is always derived from the p-state.

Denoting by E the probability per unit time of the electromagnetic transition $1p - 1s$ and by T_s^γ and T_p^γ those corresponding to T_s and T_p for the radiative absorption

$$\pi^- + \mathrm{d} \to 2\mathrm{n} + \gamma$$

the following relations are obtained:

$$f_s = \frac{E}{E + T_p + T_p^\gamma} \cdot \frac{T_s}{T_s + T^\gamma}, \tag{5}$$

$$f_p = \frac{T_p}{E + T_p + T_p^\gamma}. \tag{6}$$

([8]) C. Rubbia and M. Toller: *Nuovo Cimento*, **10**, 410 (1958). We wish to thank dr. L. di Lella for a number of useful informations on this subject.

([9]) E. Fermi and E. Teller: *Phys. Rev.*, **72**, 299 (1947).

([10]) A. S. Wightman: *Phys. Rev.*, **77**, 521 (1950).

([11]) R. A. Ferrel: *Phys. Rev. Lett.*, **4**, 425 (1960); A. Fevsner, R. Strand, L. Madansky and T. Toohig: *Nuovo Cimento*, **19**, 409 (1961).

In these equations four unknown quantities appear, *i.e.*, T_s, T_p, T_s^γ and T_p^γ, while E can be calculated with reasonable confidence. In writing eq. (5) and (6) it is taken into account that the reaction $\pi^- + d \to 2n + \pi^0$ is unimportant and that the absorption mean-life is much smaller than the pion decay mean-life.

It should be noted that the experimental ratio between the non-radiative absorption probability

$$f = f_s + f_p$$

and the radiative absorption probability (*)

$$f^\gamma = 1 - f$$

is found to be about two [1]. Unfortunately, it seems to be difficult to devise a criterion which could distinguish between the s- and p-contributions to the radiative absorption.

Eq. (5) and (6) may also be written

$$f_s S_s + f_p S_p = f^\gamma\,, \tag{7}$$

$$T_p(1 - f_p - f_p S_p) = f_p E\,, \tag{8}$$

by putting

$$S_s = \frac{T_s^\gamma}{T_s} \qquad \text{and} \qquad S_p = \frac{T_p^\gamma}{T_p}\,.$$

The measurement of f_s and f_p then gives two relations for the quantities S_s, S_p and T_p. To obtain more information one could perhaps measure the intensity of the X-rays emitted in the meson transition $1p \to 1s$. In doing so the probability of such a process, given by

$$f^X = \frac{E}{E + T_p + T_p^\gamma}\,, \tag{9}$$

could be determined.

Eq. (7), (8) and (9) may be rearranged in the following way:

$$S_s = \frac{f^X - f_s}{f_s}\,, \tag{10}$$

$$S_p = \frac{f^\gamma - f_s}{f_p}\,, \tag{11}$$

$$T_p = \frac{f_p}{f^X} \cdot E\,. \tag{12}$$

(*) The γ-rays from $\pi^- + d \to 2n + \gamma$ are nearly monoenergetic, with an energy close to the pion mass.

It is to be noted that the ratio $S_s = T_s^\gamma / T_s$ is the only obtainable information about the s-absorption from the knowledge of f_s, f_p and f^X.

In order to know T_s and T_s^γ separately, a measurement of the $1s$-state mean-life would be required. This could be done, in principle, by detecting the delayed coincidences between the X-ray, which give rise to this state, and the γ-ray of the radiative absorption. Another method could be based on the width determination for the lines $1p \to 1s$, $2p \to 1s$, etc. However all this is near the limits of the experimental possibilities.

4. – We are now going to sketch, very roughly, a possible extension to complex nuclei, following what was said in Section **2** about the non-radiative absorption in deuterium.

For this, some assumptions about the two-nucleon correlation function inside the nuclear matter are needed.

We shall make use of the so called « quasi-deuteron model » ([12]) or « independent pair model » ([13]) and we shall suppose the relative motion of the two interacting nucleons to be an s-wave ([13,14]).

Moreover the C.M. motion of the quasi-deuteron ([15]) will be neglected.

Since we are interested in the two-neutron emission, the interacting pair which absorbs the pion will be a proton-neutron pair. Therefore both the singlet and triplet interactions must be taken into consideration.

If the quasi-deuteron is in a triplet spin-state, all we have said in Section **2** holds unchanged.

We go then to the case of a quasi-deuteron in a singlet spin-state.

First of all it is to be noticed that there is no p-absorption. Indeed a two-neutron antisymmetric state, having $J = 1$ and positive parity, does not exist.

Instead, the s-absorption is possible and leads to the final state $y_{110}^{\;0}$. Limiting ourselves to the case of transverse polarizations, it is easy to verify that $P_{110} = +1$ if the axes z and z' are taken parallel and pointing in the same direction.

To summarize: the p-absorption (to which only the quasi-deuterons in a triplet state contribute) leads to antiparallel transverse polarizations of the two neutrons. The s-absorption leads to uncorrelated polarizations, if the quasi-deuteron is in a triplet state, and to parallel transverse polarizations if the quasi-deuteron is in a singlet state.

([12]) J. Heidmann: *Phys. Rev.*, **80**, 171 (1950); J. S. Levinger: *Phys. Rev.*, **84**, 43 (1951).

([13]) L. C. Gomes, J. D. Walecka and V. F. Weisskopf: *Ann. Phys.*, **3**, 241 (1958).

([14]) K. Gottfried: *Nucl. Phys.*, **5**, 557 (1958); T. Tagami: *Prog. Theor. Phys.*, **21**, 533 (1959).

([15]) L. S. Azhgirei, I. K. Vzorov, V. P. Zrelov, M. G. Mescheriakov, B. S. Neganov and A. F. Shabudin: *Soviet Phys. J.E.T.P.*, **6**, 911 (1958).

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RIASSUNTO

Si esamina la correlazione fra le polarizzazioni dei neutroni emessi nella reazione di cattura: $\pi^- + d \to 2n$. Ne risulta una semplice relazione con lo stato del mesone assorbito, ottenendo così un mezzo per distinguere i contributi al processo da parte di stati mesonici *s* e *p*. Si accenna anche brevemente ad una possibile estensione al caso di cattura in nuclei complessi.

On a Generalized Foldy-Wouthuysen Transformation.

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Summary. — A correspondence between the Lorents and the Foldy-Wouthuysen transformation is discussed. It is extended to any Lorentz transformation and it is shown that the transformation defined by Cini-Toushek fits as a special case of this correspondence. It is shown that while the Lorentz transformation operates in a space with an indefinite Lorentz invariant metric, the Foldy transformation here discussed operates in a space with a positive definite euclidean metric. This correspondence is extended to any spin. A particular discussion is given for the spin 1 case, for which a hamiltonian equation is proposed.

1. – Introduction.

In the spin $\frac{1}{2}$ case the problem of deducing a Pauli type equation (*i.e.* one without the weak components) equivalent to the Dirac one has been solved by Foldy-Wouthuysen [1]. The method used by these authors consists in finding a unitary transformation that eliminates the odd operators from the Dirac hamiltonian. In a similar way, it is possible to introduce another unitary transformation which eliminates the even part of the Dirac hamiltonian. This has been done by Cini-Toushek [2].

In this work we are going to adopt a viewpoint which stems from the analogy, pointed out by one of us [3], between the Foldy-Wouthuysen trans-

(1) L. Foldy and S. Wouthuysen: *Phys. Rev.*, **78**, 29 (1950).
(2) M. Cini and B. Toushek: *Nuovo Cimento*, **3**, 424 (1958).
(3) J. J. Giambiagi: *Nuovo Cimento*, **16**, 202 (1960).

formation [4] and the Lorentz transformation that refers the particle to its rest system. We want to discuss that analogy in a more complete way, without reference to the hamiltonian, including in it the Cini-Toushek transformation [5] and extending the analysis to higher spins, with a special discussion for the spin-one case. The inclusion of the C.T.t. within the general scheme and the possibility of its extension to higher spin values in a natural and uniform way further supports our definition for the general F.W.t. This definition should be compared with that introduced by CASE [6] for spin one The latter does not allow one to include the C.T.t. in the general formalism.

Furthermore, being independent of the hamiltonian, the F.W.t. here considered allows the introduction, by a simple analogy with the spin-$\frac{1}{2}$ case, of a hamiltonian equation of motion for higher spin particles.

2. – Spin $\frac{1}{2}$.

The Lorentz transformation is of course, the natural transformation to be used when one tries to establish the connection between the same experimental situation as described in two different Lorentz systems. This transformation leaves invariant the scalar product defined by

$$(2.1)\qquad (\varphi, \psi)_L = \overline{\varphi}\psi = \varphi^*\beta\psi \; ; \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

where φ^* is related to φ by a coniugation and a transposition.

On the other hand, the F.W.t. is not unitary within the scalar product defined by (2.1). Instead, it is unitary with the positive definite scalar product,

$$(2.2)\qquad (\varphi, \psi)_E = \varphi^*\psi \, .$$

The latter is a more suitable definition for the hamiltonian representation of the movement of a particle, because the Dirac hamiltonian is hermitian with respect to (2.2) but not with respect to (2.1).

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[4] Hereafter referred to as F.W.t.
[5] Hereafter referred to as C.T.t.
[6] R. CASE: *Phys. Rev.*, **96**, 1323 (1954).

We write the Dirac hamiltonian

$$\mathscr{H} = \alpha \cdot p + \beta m = \beta_\mu q_\mu \tag{2.3}$$

$$\left.\begin{aligned} \beta_i &= \alpha_i \\ q_i &= p_i \end{aligned}\right\} \quad i = 1, 2, 3 \,. \qquad \begin{aligned} \beta_4 &= \beta \\ q_4 &= m \end{aligned} \quad ; \qquad \beta_\mu^* = \beta_\mu ,$$

$$\mathscr{H}^* = \mathscr{H} \,. \tag{2.4}$$

Suppose now for a while, that we did not know the form of the hamiltonian (2.3) and accordingly we did not know the form of the spinor representing a particle with impulse $\boldsymbol{p}$. It is nevertheless possible to obtain that spinor if we adopt the following procedure.

Let us start from a particle at rest, described by a spinor ψ_0, having two components and containing the state of polarization and the sign of the energy. For the representation of a particle with a certain momentum $\boldsymbol{p}$, one can use the spinor (with the appropriate exponential factor)

$$\psi(p) = L_p^{-1} \psi_0 \,, \tag{2.5}$$

where L_p is the Lorentz transformation that reduces p to zero

$$L_p = \frac{E + m - \alpha \cdot p}{\sqrt{2m(E+m)}} \,. \tag{2.6}$$

However, as we said, this transformation is not suitable for a hamiltonian representation because it does not leave invariant the positive definite scalar product (2.2). In other words, L is not unitary within (2.2). Nevertheless, a little « amendment » or correction will make L unitary.

We first observe that, if the energy is positive

$$L_p^{-1} \psi_0 = \frac{E + m + \alpha \cdot p}{\sqrt{2m(E+m)}} \psi_0 = \frac{E + m + \alpha \cdot p \beta}{\sqrt{2m(E+m)}} \psi_0 \,,$$

because $\beta \psi_0 = \psi_0$. If the energy is negative

$$L_p^{-1} = \frac{E + m - a \cdot p}{\sqrt{2m(E+m)}} \,,$$

$$L_p^{-1} \psi_0 = \frac{E + m - \alpha \cdot p}{\sqrt{2m(E+m)}} \psi_0 = \frac{E + m + \alpha \cdot p \beta}{\sqrt{2m(E+m)}} \psi_0 \,,$$

because $\beta \psi_0 = -\psi_0$.

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Lastly, if one wants $\psi^*\psi = \psi_0^*\psi_0$, one must multiply by the «renormalization» factor $\sqrt{m/E}$. After that

$$\psi = \frac{E + m + \alpha\cdot p\beta}{\sqrt{2E(E+m)}}\,\psi_0\,. \tag{2.7}$$

And the transformation is now unitary within (2.2) and in fact coincides with the inverse of the F.W.t.

$$F_p^{-1} = \frac{E + m + \alpha\cdot p\beta}{\sqrt{2E(E+m)}}\,. \tag{2.8}$$

We have established in that way a correspondence between the Lorentz transformation that takes the particle to its rest system and the F.W.t. We can now say that the latter allows one to describe the polarization of the particle (within the hamiltonian formulation) by means of a «spinor at rest».

The correspondence can easily be extended. Any Lorentz transformation that transforms p_1 into p_2 can be expressed as

$$L(p_1/p_2) = L_{p_2}^{-1}\cdot L_{p_1}\,. \tag{2.9}$$

Which leads us to the definition

$$F(p_1/p_2) = F_{p_2}^{-1} F_{p_1}\,. \tag{2.10}$$

Thus, the correspondence is completed.

Furthermore, we can now give a sense to a transformation to the velocity of light. In spite of the fact that $L\,(p_1/\infty)$ is non-existent because L_p blows up in the limit $p\to\infty$, the corresponding limit of the induced F.W.t. has a perfectly definite sense.

$$F_\infty = \frac{1 + \beta\alpha\cdot \boldsymbol{C}}{\sqrt{2}}\,. \tag{2.11}$$

where $\boldsymbol{C}$ is a unit vector in the direction of $\boldsymbol{p}$. (2.11) can be deduced from

$$F_p = \frac{E + m + \beta\alpha\cdot\boldsymbol{p}}{\sqrt{2E(E+m)}}\,, \tag{2.12}$$

if we notice that for $E\to\infty$, $|p|/|E|\to 1$.

A F.W.t. to the velocity of light is then

$$C_p = F(p/\infty) = F_\infty^{-1} F_p\,. \tag{2.13}$$

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An actual calculation, *i.e.*, a multiplication of the inverse of (2.11) by (2.12) gives

$$C_p = \frac{E + p - (m/p)\beta\alpha\cdot p}{\sqrt{2E(E+p)}} \,. \tag{2.14}$$

And this is precisely the C.T.t., which can then be interpreted as a F.W.t. to the velocity of light.

To end this chapter we shall write these transformations in the usual more compact exponential form

$$L_p = \exp\left[\frac{\alpha\cdot p}{2p}\,\mathrm{arctgh}\,\frac{p}{E}\right], \tag{2.15}$$

$$\begin{cases} F_p = \exp\left[-\beta\,\dfrac{\alpha\cdot p}{2p}\,\mathrm{arctg}\,\dfrac{p}{m}\right], \\[2ex] C_p = F_\infty^{-1}\,F_p = \exp\left[\beta\,\dfrac{\alpha\cdot p}{2p}\left[\dfrac{\pi}{2} - \mathrm{arctg}\,\dfrac{p}{m}\right]\right] = \exp\left[\beta\,\dfrac{\alpha\cdot p}{2p}\,\mathrm{arcotg}\,\dfrac{p}{m}\right], \end{cases} \tag{2.16}$$

$$C_p = \exp\left[\beta\,\frac{\alpha\cdot p}{2p}\,\mathrm{arctg}\,\frac{m}{p}\right]. \tag{2.17}$$

3. – Spin 1.

We know that, under a Lorentz transformation

$$x'_\mu = a_\mu{}^\nu x_\nu\,; \qquad a_\mu{}^\nu a^\mu{}_e = \delta_e{}^\nu\,, \tag{3.1}$$

any four vector φ_μ changes to

$$\varphi'_\mu = a_\mu{}^\nu \varphi_\nu\,. \tag{3.2}$$

On account of (3.1), this transformation leaves invariant the scalar product

$$\varphi^*_\mu \varphi^\mu = \varphi^*_\mu g^{\mu\nu} \varphi_\nu \qquad [g^{ij} = \delta^{ij}\,;\ i, j = 1, 2, 3\,;\ g^{44} = -1]. \tag{3.3}$$

(3.3) should be compared with (2.1) where we see that β plays the role of a metric. Sometimes, to stress further that analogy, we shall write (3.3) with a matrix notation

$$(\varphi, \psi)_L = \varphi^* g \psi\,. \tag{3.4}$$

The Lorentz spinor transformation is not independent from the vectorial

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transformation. We have

$$L^{-1}\gamma_\mu L = a_\mu{}^\nu \gamma_\nu \tag{3.5}$$

which is a consequence of the relativistic invariance of the Dirac equation.

From (3.5) we can deduce

$$a_{\mu\nu} = \tfrac{1}{4}\,\mathrm{Tr}\,\{L^{-1}\gamma_\mu L\gamma_\nu\}\,, \tag{3.6}$$

with

$$\tfrac{1}{4}\,\mathrm{Tr}\,\{\gamma_\mu\gamma_\nu\} = g_{\mu\nu}\,.$$

Eq. (3.6) could be used to find out the coefficients $a_{\mu\nu}$ from L_p, if they had not been known beforehand

$$\begin{aligned} a_{\mu\nu}(p) = g_{\mu\nu} + \frac{1}{m}\,(g_{4\mu}g_{i\nu} - g_{i\mu}g_{4\nu})\,p^i\,+ \\ + \left(\frac{E}{m} - 1\right)\left(g_{i\mu}\,g_{j\nu}\,\frac{p^i\,p^j}{p^2} - g_{4\mu}\,g_{4\nu}\right). \qquad (i, j = 1, 2, 3). \end{aligned} \tag{3.7}$$

It should be kept in mind that the Lorentz transformation leaves invariant the relativistic form of the Dirac equation

$$i\gamma_\mu p^\mu \psi = m\psi\,; \qquad \gamma_i = i\beta\alpha_i\,; \qquad \gamma_4 = i\beta\,, \tag{3.8}$$

while the F.W.t. leaves invariant the hamiltonian form of the Dirac equation (see (2.3))

$$\beta_\mu q_\mu \psi = E\psi\,. \tag{3.9}$$

We are now looking for a four vector transformation, analogous to $a_{\mu\nu}$, but related to the F.W.t. in a similar way to that in which $a_{\mu\nu}$ is related to the Lorentz transformation, as shown by (3.6).

A look at the eq. (3.8) and (3.9) strongly suggests a change of β_μ for γ_μ to be simultaneously made with that of F for L in (3.6). In this way we are guided to the following *definition* for the spin-1 F.W.t.

$$b_{\mu\nu}(p) = \tfrac{1}{4}\,\mathrm{Tr}\,\{F_p^{-1}\beta_\mu F_p\beta_\nu\}\,, \tag{3.10}$$

with

$$\tfrac{1}{4}\,\mathrm{Tr}\,\{\beta_\mu\beta_\nu\} = \delta_{\mu\nu}\,.$$

An actual calculation gives

$$b_{\mu\nu}(p) = \delta_{\mu\nu} + \frac{1}{E}\,(\delta_{4\mu}\,\delta_{i\nu} - \delta_{4\nu}\,\delta_{i\mu})\,p_i + \left(\frac{m}{E} - 1\right)\left(\delta_{i\mu}\,\delta_{j\nu}\,\frac{p^i\,p^j}{p^2} + \delta_{4\mu}\delta_{4\nu}\right). \tag{3.11}$$

It is easy to see that

$$b_{\mu\nu} b_{\mu\varrho} = b_{\nu\mu} b_{\varrho\mu} = \delta_{\nu\varrho} \,. \tag{3.12}$$

So that the definition (3.10) implies that the transformation $b_{\mu\nu}$ leaves invariant the euclidean positive definite scalar product

$$\varphi_\mu^* \psi_\mu \,. \tag{3.13}$$

So we have the same situation as in spin $\frac{1}{2}$. The F.W.t. leaves invariant a positive definite scalar product and the Lorentz transformation an indefinite one.

A further comparison of (3.7) and (3.11) shows that a Lorentz transformation to the rest system brings p_μ to the form

$$p'_\mu = a_\mu^{\ \nu} p_\nu \,; \qquad p' = (0, 0, 0, m) \tag{3.14}$$

leaving invariant the modulus $p_\mu p^\mu = p^2 - E^2 = m^2$.

While the F.W.t (3.11) has the property that q_μ (cf. (2.3)) is brought to the form

$$q'_\mu = b_{\mu\nu} q_\nu \,; \qquad q' = (0, 0, 0, E) \tag{3.15}$$

and the invariant square modulus now is $q_\mu q_\mu = p^2 + m^2 = E^2$.

The F.W.t. defined by (3.10) can be extended, using (2.10) to any transformation $p_1 \to p_2$. The general F.W.t. will be, according to (2.10)

$$b_{\mu\nu}(p_1/p_2) = b^{-1}_{\mu\varrho}(p_2)\, b_{\varrho\nu}(p_1)$$

and we can easily deduce

$$b_{\mu\nu}(p_1/p_2) = \tfrac{1}{4} \operatorname{Tr}\{F^{-1}(p_1/p_2)\beta_\mu F(p_1/p_2)\beta_\nu\}$$

which generalizes (3.10).

As in the spin-$\frac{1}{2}$ case, the limit F_p for $p \to \infty$ has a definite meaning.

$$b_{\mu\nu}(\infty) = \delta_{\mu\nu} + (\delta_{4\mu}\delta_{i\nu} - \delta_{4\nu}\delta_{i\mu})c_i - \delta_{i\mu}\delta_{j\nu}c_i c_j - \delta_{4\mu}\delta_{4\nu} \,. \tag{3.16}$$

And a C.T.t. can again be defined as the F.W.t. to the velocity of light (cf. (2.13))

$$C_p = F^{(1)-1}_\infty F^{(1)}_p \,, \tag{3.17}$$

with coefficients

$$c_{\mu\nu} = b^{-1}_{\mu\varrho}(\infty)\, b_{\varrho\nu}(p) \,.$$

From (3.11) and 3(.16) we obtain

$$c_{\mu\nu} = \delta_{\mu\nu} + \frac{m}{E}(\delta_{i\mu}\delta_{4\nu} - \delta_{4\mu}\delta_{i\nu})\,p^i + \left(\frac{p}{E} - 1\right)\left(\delta_{i\mu}\delta_{j\nu}\frac{p_i p_j}{p^2} + \delta_{4\mu}\delta_{4\nu}\right). \tag{3.18}$$

It can also be verified that

$$c_{\mu\nu} = \tfrac{1}{4}\,\mathrm{Tr}\{C\beta_\mu C^{-1}\beta_\nu\}\,,$$

as was to be expected because C is a special F.W.t.

The F.W.t. defined by (3.10) and the C.T.t. defined by (3.17) can also be obtained through another analogy with the spin-$\frac{1}{2}$ case.

Let us write the Lorentz transformation $a_\mu^{\ \nu}$ in the exponential form

$$L^{(1)} = \exp\left[-\,iS^{i4}\frac{v_i}{v}\,\mathrm{arctgh}\,v\right]. \tag{3.19}$$

where

$$S^{i4}_{\mu\nu} = \frac{1}{i}(\delta^i_\mu\delta^4_\nu - \delta^i_\nu\delta^4_\mu)\,.$$

If we compare the exponential forms for L and F (*) given in Section **2** (2.15) and (2.16), we notice that we can pass from the former to the latter by the following procedure. First, multiply S^{i4} (in (2.15)) by β on the left. Second, change arctgh v into arctg p/m. Now, keeping in mind that for spin 1 g is the analogous of β, we get from (3.19)

$$\exp\left[-\,igS^{i4}\frac{v_i}{v}\,\mathrm{arctg}\,\frac{p}{m}\right] = F^{(1)}\,, \tag{3.20}$$

which is the F.W.t. already defined by (3.10). The exponential form for the spin-1 C.T.t. can now be determined from the general definition (3.17)

$$C^{(1)} = \exp\left[igS^{i4}\frac{v_i}{v}\,\mathrm{arctg}\,\frac{m}{p}\right]. \tag{3.21}$$

4. – Hamiltonian equation for spin one.

The adoption of the definition (3.11) for the coefficients of the F.W.t. implies the existence of the following relation

$$b_{\mu\nu}\varphi_\nu(p) = \varphi_\mu(0)\,. \tag{4.1}$$

(*) When no super-index is used, the corresponding transformation for spin $\frac{1}{2}$ is meant.

Now, the vector « at rest » has its fourth component equal to zero, *i.e.* it fulfils the condition

$$g_{\mu\nu}\varphi_\nu(0) = \varphi_\mu(0)\,. \tag{4.2}$$

Using (4.1), (4.2) can be written

$$\begin{cases} g_{\mu\nu}b_{\nu\varrho}\varphi_\varrho(p) = b_{\mu\varrho}\varphi_\varrho(p) \\ b^{-1}_{\sigma\mu}g_{\mu\nu}b_{\nu\varrho}\varphi_\varrho(p) = \varphi_\sigma(p) \end{cases} \tag{4.2$'$}$$

or, in matrix notation

$$F^{(1)-1}gF^{(1)}\varphi = \varphi\,. \tag{4.3}$$

We may now recall that for spin $\frac{1}{2}$ the hamiltonian form of the Dirac equation of motion can be written in the form

$$F^{-1}\beta Fq\varphi = E\varphi\,, \tag{4.4}$$

$$q = \sqrt{q_\mu q_\mu} = \sqrt{m^2 + p^2}\,,$$

because

$$F^{-1}\beta F = \frac{\beta m - \alpha\cdot p}{q} = \frac{\beta_\mu q_\mu}{q}\,. \tag{4.5}$$

The analogous of the last expression for spin 1 is $F^{(1)-1}gF^{(1)}$ which is equal to

$$F^{(1)-1}gF^{(1)} = \frac{\beta_{\mu\nu}q_\mu q_\nu}{q^2}\,, \tag{4.6}$$

with

$$\beta^{\varrho\sigma}_{\mu\nu} = \delta_{\mu\nu}\delta^{\varrho\sigma} - \delta^\varrho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\varrho_\nu\,. \tag{4.7}$$

I.e.

$$\beta^{\varrho\sigma}_{44} = g^{\varrho\sigma}\,; \qquad \beta^{\varrho\sigma}_{4i} + \beta^{\varrho\sigma}_{i4} = -2\delta^\varrho_4\delta^\sigma_i - 2\delta^\varrho_i\delta^\sigma_4\,,$$

$$\beta^{\varrho\sigma}_{ij} = -\delta^\varrho_i\delta^\sigma_j - \delta^\sigma_i\delta^\varrho_j\,.$$

A comparison of (4.5) and (4.6) shows that, if for spin $\frac{1}{2}$ the hamiltonian is taken to be

$$\mathscr{H} = qF^{-1}\beta F = \beta_\mu q_\mu$$

the analogous expression for spin 1 must be

$$\mathscr{H}^2 = q^2F^{(1)-1}gF^{(1)} = \beta_{\mu\nu}q_\mu q_\nu\,. \tag{4.8}$$

And the « hamiltonian » equation of motion is now

$$(4.9)\qquad \begin{cases} \mathscr{H}^2\psi = E^2\psi\,, \\ \beta_{\mu\nu}q_\mu q_\nu\psi = E^2\psi\,. \end{cases}$$

Or more explicitly from (4.7)

$$(4.10)\qquad (q^2\delta_{\varrho\sigma} - 2q_\varrho q_\sigma)\psi_\sigma = E^2\psi_\varrho\,.$$

It is easy to show from (4.10) that

$$\mathscr{H}^2\cdot\mathscr{H}^2 = q^4$$

so that (4.9) implies

$$(4.11)\qquad q^2\psi = E^2\psi$$

which is the « hamiltonian » form of the Klein-Gordon equation.

A multiplication by q_ϱ changes (4.10) into

$$(4.12)\qquad (q_\sigma q^2 - 2q_\sigma q^2)\psi_\sigma = E^2 q_\sigma\psi_\sigma \qquad i.e. \qquad q_\sigma\psi_\sigma = 0\,.$$

We may observe that with this formulation, the relativistically invariant Lorentz condition is replaced by the euclidean invariant expression (4.12) which we call the Foldy condition. The Foldy condition (4.12) implies that in the rest system $\psi_4 = 0$.

5. – Extension to higher spin.

The analogy between the Lorentz transformation and the general F.W.t. allows us to extend the definition of the latter for higher spin.

The particle is in general represented by a tensor of the n-th rank where either $n = s$ for integer spin or $n = s - \frac{1}{2}$ for half integer spin. In the last case the wave function is also a spinor of the first degree (7).

Under a Lorentz transformation one has

$$(5.1)\qquad \psi'_{\mu_1\ldots\mu_n} = a_{\mu_1}{}^{\nu_1}\ldots a_{\mu_n}{}^{\nu_n} L\psi_{\nu_1\ldots\nu_n}\,,$$

where the L is to be suppressed when $n = s$.

(7) W. Rarita and J. Schwinger: *Phys. Rev.*, **60**, 61 (1941).

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In matrix notation

$$\psi' = L^{(s)}\psi\,; \qquad L^{(s)} = \underbrace{L^{(1)}\times L^{(1)}\times\ldots\times L^{(1)}\times L}_{n+1 \text{ factors}}\,. \tag{5.2}$$

The symbol $L^{(1)}\times L^{(1)}$ meaning the outer product of two vectorial Lorentz transformations.

The general Lorentz transformation (5.1) leads us immediately to the following general definition valid for any spin

$$\psi'' = F^{(s)}\psi = \underbrace{F^{(1)}\times F^{(1)}\times\ldots\times F^{(1)}\times F}_{n+1 \text{ factors}}\,\psi\,. \tag{5.3}$$

Or, more explicitly

$$\psi''_{\mu_1\ldots\mu_n} = b_{\mu_1\nu_1}\ldots b_{\mu_n\nu_n}\cdot F\psi_{\nu_1\ldots\nu_n}\,. \tag{5.4}$$

The same formal expression is valid for the C.T.t. but this is implicit in our interpretation of the latter as a special case of the general F.W.t.

A general exponential form can be written for the Lorentz transformation and this is also true for the F.W.t.

Taking the spin-2 case as an example, we have

$$F^{(2)} = \exp\left[[igS^{(1)}_{i4}\frac{v_i}{v}\operatorname{arctg}\frac{p}{m}\right]\cdot\exp\left[igS^{(1)}_{i4}\frac{v_i}{v}\operatorname{arctg}\frac{p}{m}\right], \tag{5.5}$$

$$F^{(\cdot)} = \exp\left[i[(gS^{(1)}_{i4})_1\;\delta_2]\frac{v_i}{v}\operatorname{arctg}\frac{p}{m}\right]\cdot\exp\left[i[\delta_1(gS_{i4})_2]\frac{v_i}{v}\operatorname{arctg}\frac{p}{m}\right]. \tag{5.6}$$

The sub-indices 1 and 2 refer to the first or second of the vectorial indices of the wave function. δ_1 is the unit matrix with respect to the first index and δ_2 is the unit matrix with respect to the second one.

From (5.6) we obtain

$$F^{(2)} = \exp\left[i[(gS^{(1)}_{i4})_1\;\delta_2 + \delta_1(gS^{(1)}_{i4})_2]\frac{v_i}{v}\operatorname{arctg}\frac{p}{m}\right],$$

$$F^{(2)} = \exp\left[ig_1g_2[(S^{(1)}_{i4})_1g_2 + g_1(S^{(1)}_{i4})_2]\frac{v_i}{v}\operatorname{arctg}\frac{p}{m}\right],$$

because $g\cdot g = \delta$; $g_{\mu\nu}g_{\nu\varrho} = \delta_{\mu\varrho}$.

Using the notation G for $g_1 g_2$, and noting that

$$(S_{i4}^{(1)})_1 g_2 + g_1 (S_{i4}^{(1)})_2 = (S_{i4}^{(2)})_{12} ,$$

we have

$$F^{(2)} = \exp \left[i G S_{i4}^{(2)} \frac{v_i}{v} \operatorname{arctg} \frac{p}{m} \right] .$$

In the general case, the F.W.t. that refers the particle to its rest system is

$$F^{(s)} = \exp \left[i G^{(s)} S_{i4}^{(s)} \frac{v_i}{v} \operatorname{arctg} \frac{p}{m} \right] , \tag{5.7}$$

with

$$G^{(s)} = \underbrace{g \times g \times \ldots \times g \times \beta}_{n+1 \text{ factors}} , \tag{5.8}$$

where the β matrix must be suppressed in the integer spin cases.

* * *

One of the authors (J.J.G.) is indebted to Prof. J. Tiomno for many useful discussions on this suject.

RIASSUNTO (*)

Discutiamo la corrispondenza fra la trasformazione di Lorentz e quella di Foldy-Wouthuysen. La estendiamo ad ogni trasformazione di Lorentz e mostriamo che la trasformazione definita da Cini-Touschek si inserisce come un caso speciale di questa corrispondenza. Mostriamo che mentre la trasformazione di Lorentz opera in uno spazio con una metrica indefinita, invariante per trasformazioni di Lorentz, la trasformazione di Foldy, che qui discutiamo, opera in uno spazio avente una metrica euclidea positiva definita. Questa corrispondenza viene estesa a tutti gli spin. Diamo una particolare discussione per il caso di spin 1, per il quale si propone un'equazione hamiltoniana.

(*) *Traduzione a cura della Redazione.*

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Contribution of Neutral Pions to Photon-Photon Scattering (*).

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Summary. — It is shown that the contribution of neutral pion intermediate states is as much important as electron-positron pair states for low energy photon-photon scattering. For unpolarized scattering the interference between two intermediate states is shown to vanish and the contribution of neutral pion states to the differential cross section is shown to have a sharp maximum with $1.15 \cdot 10^{-26}$ cm²/sr at a total energy of photons corresponding to the pion mass. The forward and right-angle cross sections for two intermediate states are compared.

1. – Introduction.

In view of the success of the quantum electrodynamics (QED) in the quantitative explanation of the Lamb shift and the anomalous magnetic moment of the electron, it is generally believed that we have a reliable method to predict the behavior of electrons and photons if we ignore their interaction with other particles. But actually a photon can interact with any particle that has a charge and/or a magnetic moment; and, further, we know that a neutral pion decays into two photons. The existence of these interactions is expected to modify the result of QED, especially at energies above the threshold of the production of particles heavier than the electron.

In this paper, we will report on the correction to photon-photon scattering due to the existence of particles other than the electron, in particular, the neutral pion. This process is particularly interesting because the correction

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is of the same order in the electric charge (*i.e.* e^4) as the lowest order term of the perturbation expansion in QED, although the relevant corrections to electron-photon and electron-electron scattering are all of higher order.

2. – Matrix element for low energy photons.

The S-matrix element for the photon-photon scattering can be expressed as

(1) $$S_{fi} = \delta_{fi} + (2\pi)^4\delta^{(4)}(k_1+k_2-k_3-k_4)[(2\pi)^{12}16k_{10}k_{20}k_{30}k_{40}]^{-\frac{1}{2}}M_{fi}\,,$$

and

(2) $$M_{fi} = \sum M^{(i)}(s,t,u)\,O^{(i)}_{\mu\nu\lambda\varrho}(k_1,k_2,k_3,k_4)\,\varepsilon^{(1)}_\mu\,\varepsilon^{(2)}_\nu\,\varepsilon^{(3)}_\lambda\,\varepsilon^{(4)}_\varrho\,,$$

where $O^{(i)}_{\mu\nu\lambda\varrho}$ are tensors of the fourth rank that can be constructed from the four-momenta k_1, k_2 and k_3, k_4 of the initial and final photons respectively. and satisfy the gauge conditions

(3) $$O^{(i)}_{\mu\nu\lambda\varrho}k_{1\mu} = 0\,, \qquad O^{(i)}_{\mu\nu\lambda\varrho}k_{2\nu} = 0\,, \quad \text{etc.}$$

$\varepsilon^{(i)}_\mu$ is a polarization four-vector of the photon with momentum k_i, and $M^{(i)}(s,t,u)$ are scalar functions of $s=(k_1+k_2)^2$, $t=(k_1-k_3)^2$ and $u=(k_1-k_4)^2$ which satisfy the relation $s+t+u=0$.

Now we assume that the $M^{(i)}$'s satisfy a double dispersion relation of the following form:

(4) $$M^{(i)}(s,t,u) = \frac{A^{(i)}}{m_\pi^2-s}+\frac{B^{(i)}}{m_\pi^2-t}+\frac{C^{(i)}}{m_\pi^2-u}+ \int\frac{\varrho_s^{(i)}(s')}{s'-s}\,\mathrm{d}s'+\int\frac{\varrho_t^{(i)}(t')}{t'-t}\,\mathrm{d}t'+\int\frac{\varrho_u^{(i)}(u')}{u'-u}\,\mathrm{d}u'+\iint\frac{\varrho_{st}^{(i)}(s',t')}{(s'-s)(t'-t)}\,\mathrm{d}s'\,\mathrm{d}t'+ \iint\frac{\varrho_{tu}^{(i)}(t',u')}{(t'-t)(u'-u)}\,\mathrm{d}t'\,\mathrm{d}u'+\iint\frac{\varrho_{us}^{(i)}(u',s')}{(u'-u)(s'-s)}\,\mathrm{d}u'\,\mathrm{d}s'\,.$$

The pole terms are contributions from one pion intermediate states and residues are related to the π^0 lifetime in a simple way. Indeed

(5) $$M_\pi = \sum M_B^{(i)}\,O^{(i)}_{\mu\nu\lambda\varrho}\,\varepsilon^{(1)}_\mu\,\varepsilon^{(2)}_\nu\,\varepsilon^{(3)}_\lambda\,\varepsilon^{(4)}_\varrho$$

(where $M_B^{(i)}$ are the sum of the three pole terms) is a Born approximation to photon-photon scattering calculated from a local interaction Hamiltonian for neutral pion decay of the form $f\varepsilon_{\mu\nu\lambda\varrho}F_{\mu\nu}F_{\lambda\varrho}\varphi$, where f is the coupling constant, $F_{\mu\nu}$ is the electromagnetic field four-tensor and φ is a neutral pion field.

The contribution to the cross-section from the pole terms becomes infinite at $s = m_\pi^2$ corresponding to the fact that the pion mass is larger than the threshold of photon-photon scattering. But actually since the neutral pion is unstable the position of the pole is displaced by a complex quantity Δs. The imaginary part of Δs is related to the lifetime τ of π^0 by [1]

$$\mathrm{Im}\,(\Delta s) = \frac{m_\pi}{\tau}\,, \tag{6}$$

and the real part of Δs is negligible compared to m_π because the decay is of electromagnetic origin.

Thus we have the Born term of the following form:

$$M_\pi = f^2 \left[\frac{\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\lambda\varrho\omega\sigma}}{m_\pi^2 - s + i(m_\pi/\tau)} + \frac{\varepsilon_{\mu\lambda\alpha\omega}\varepsilon_{\nu\varrho\beta\sigma}}{m_\pi^2 - t + i(m_\pi/\tau)} + \frac{\varepsilon_{\mu\varrho\alpha\sigma}\varepsilon_{\nu\lambda\beta\omega}}{m_\pi^2 - u + i\,(m_\pi/\tau)}\right] \cdot \\ \cdot\, \varepsilon_\mu^{(1)}\varepsilon_\nu^{(2)}\varepsilon_\lambda^{(3)}\varepsilon_\varrho^{(4)} k_{1\alpha}k_{2\beta}k_{3\omega}k_{4\sigma}\,. \tag{7}$$

The lower limits of the single integrals in eq. (4) are $(3m_\pi)^2$ if we retain only the contribution of the order e^4. Therefore at low energy these terms can be neglected. In the same approximation the contribution of strongly interacting particles to the double integrals starts at $(2m_\pi)^2$ and below this point the contribution comes only from two-electron and two-muon states, which is just the contribution from fourth order perturbation diagrams with electron and muon internal lines.

Therefore for incident photons with energy not too much larger than half the pion mass in c.m. system, the main part of the matrix element consists of two parts: pion pole terms M_π given by eq. (7), and the contribution from the electron square diagram M_e which was calculated by KARPLUS and NEUMAN [2]. The main feature of their result will be compared with pion pole terms in Section **4**.

3. – Differential cross-section.

The cross-section for the unpolarized case can be calculated from the matrix element obtained in the last section using the following well-known formula:

$$d\sigma = (2\pi)^2 \frac{k_{10}k_{20}}{|k_1 \cdot k_2|} \int d\boldsymbol{k}_3\, d\boldsymbol{k}_4\, \delta^{(4)}(k_1 + k_2 - k_3 - k_4)\, \frac{1}{4} \sum_{\mathrm{pol}} |M_\pi + M_e|^2\,, \tag{8}$$

[1] S. DESER, M. L. GOLDBERGER, K. BAUMAN and W. THIRRING: *Phys. Rev.*, **96**, 774 (1954); see also P. T. MATTHEWS and A. SALAM: *Phys. Rev.*, **112**, 283 (1958); **115**, 1079 (1959).

[2] R. KARPLUS and M. NEUMAN: *Phys. Rev.*, **80**, 380 (1950); **83**, 776 (1951).

where the summation is over polarizations of all photons. First we shall show that the interference term $\mathrm{Re}\,(M_\pi M_e^*)$ vanishes for unpolarized scattering, and hence that the cross-section can be expressed as a sum of two terms *i.e.* the contribution from one-pion state $d\sigma_\pi$ (*) and that from the two-electron states $d\sigma_e$, where $d\sigma_\pi$ and $d\sigma_e$ are obtained from eq. (8) by replacing $M_\pi - M_e$ by M_π and M_e respectively.

For this purpose we use the following property of $M_e^{(2)}$: if a tensor of fourth rank $G_{\mu\nu\lambda\varrho}(k_1, k_2, k_3, k_4)$ is defined by

$$M_e = \varepsilon_\mu^{(1)} \varepsilon_\nu^{(2)} \varepsilon_\lambda^{(3)} \varepsilon_\varrho^{(4)} G_{\mu\nu\lambda\varrho}(k_1, k_2, k_3, k_4) \tag{9}$$

it consists of terms which are proportional to $k_{i\mu}k_{j\nu}k_{l\lambda}k_{m\varrho}$ with $i, j, l, m = 1, 2, 3$, or 4. Using eq. (7) and (9), we have the following expression for the interference term after summation over polarizations:

$$\sum_{\mathrm{pol}} \mathrm{Re}\,(M_\pi M_e^*) = \mathrm{Re}\, f^2 \left[\frac{\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\lambda\varrho\omega\sigma}}{m_\pi^2 - s + i(m_\pi/\tau)} + \frac{\varepsilon_{\mu\lambda\alpha\omega}\varepsilon_{\nu\varrho\beta\sigma}}{m_\pi^2 - t + i(m_\pi/\tau)} + \right.$$
$$\left. + \frac{\varepsilon_{\mu\varrho\alpha\sigma}\varepsilon_{\nu\lambda\varrho\omega}}{m_\pi^2 - u + i(m_\pi/\tau)}\right] \cdot k_{1\alpha}k_{2\beta}k_{3\omega}k_{4\sigma}G_{\mu\nu\lambda\varrho}\,. \tag{10}$$

Therefore, due to the above-mentioned property of $G_{\mu\nu\lambda\varrho}$ the interference term can be broken up into terms each of which has a factor of the following form:

$$\varepsilon_{\mu\nu\lambda\omega} k_{i\mu} k_{j\nu} k_{a\lambda} k_{b\omega}\, \varepsilon_{\varrho\sigma\alpha\beta} k_{l\varrho} k_{m\sigma} k_{c\alpha} k_{d\beta}\,. \tag{11}$$

The expression (11) vanishes due to the antisymmetry of $\varepsilon_{\mu\nu\lambda\omega}$ if any two of the $(ijab)$ are the same or if any two of the $(lmcd)$ are the same, and all the remaining terms are of the form

$$[\varepsilon_{\lambda\mu\nu\omega} k_{1\lambda} k_{2\mu} k_{3\nu} k_{4\omega}]^2\,, \tag{12}$$

which is also zero. This can be seen most easily in the center-of-mass system, where $\boldsymbol{k}_1 = -\boldsymbol{k}_2$, $\boldsymbol{k}_3 = -\boldsymbol{k}_4$ and all the photons have the same energy ω. The expression in the bracket in eq. (12) consists of terms of the form

$$\varepsilon_{\lambda\mu\nu 0} k_{i\lambda} k_{j\mu} k_{l\nu}\omega = \omega\, \boldsymbol{k}_i \cdot (\boldsymbol{k}_j \times \boldsymbol{k}_l) = 0\,. \tag{13}$$

(*) *Note added in proof.* – After submission of this paper, we learned that a similar result for $d\sigma_\pi$ was obtained by V. N. ORAEVSKII: *Žurn. Ėksp. Teor. Fiz.*, **39**, 1049 (1960); *JEPT*, **12**, 730 (1961).

The last step follows since we have only two linearly independent three-momenta. This completes the proof that the interference between pole term and square diagram vanishes after the appropriate polarization summations.

The calculation of $d\sigma_\pi$ is straightforward. From (7) we have, after summation over polarization

$$\sum_{\text{pol}} |M_\pi|^2 = |M_s|^2 (k_1\cdot k_2)^4 + |M_t|^2 (k_1\cdot k_3)^4 + |M_u|^2 (k_1\cdot k_4)^4 + \\ + \operatorname{Re}(M_s M_t^*)(k_1\cdot k_2)^2 (k_1\cdot k_3)^2 + \operatorname{Re}(M_s M_u^*)(k_1\cdot k_2)^2 (k_1\cdot k_4)^2 + \\ + \operatorname{Re}(M_t M_u^*)(k_1\cdot k_3)^2 (k_1\cdot k_4)^2 , \tag{12}$$

where

$$M_s = \frac{f^2}{m_\pi^2 - s + i(m_\pi/\tau)} ,$$

and likewise for M_t and M_u.

In the center-of-mass system, we have

$$k_1\cdot k_2 = 2\omega^2 ,$$

$$k_1\cdot k_3 = \omega^2(1-\cos\theta) ,$$

$$k_1\cdot k_4 = \omega^2(1+\cos\theta) ,$$

where θ is the scattering angle. Therefore we obtain for $d\sigma_\pi/d\Omega$

$$\frac{d\sigma_\pi}{d\Omega} = \frac{f^4}{2^6(2\pi)^2}\frac{\omega^6}{m_\pi^4}\left\{\frac{16}{F^2+\Gamma^2} + \frac{(1-\cos\theta)^4}{G^2+\Gamma^2} + \frac{(1+\cos\theta)^4}{K^2+\Gamma^2} - \right. \\ \left. - \frac{4(1-\cos\theta)^2(FG+\Gamma^2)}{(F^2+\Gamma^2)(G^2+\Gamma^2)} - \frac{4(1+\cos\theta)^2(FK+\Gamma^2)}{(F^2+\Gamma^2)(K^2+\Gamma^2)} + \frac{(1-\cos^2\theta)^2(GK+\Gamma^2)}{(G^2+\Gamma^2)(K^2+\Gamma^2)}\right\} , \tag{13}$$

with

$$F = \frac{4\omega^2}{m_\pi^2} - 1 , \quad G = 1 + \frac{2\omega^2}{m_\pi^2}(1-\cos\theta) ,$$

$$K = 1 + \frac{2\omega^2}{m_\pi^2}(1+\cos\theta) , \quad \text{and} \quad \Gamma = \frac{1}{m_\pi \tau} .$$

The coupling constant f is related to the lifetime τ of π^0 by

$$\frac{1}{\tau} = \frac{m_\pi^3 f^2}{2^5\pi} . \tag{14}$$

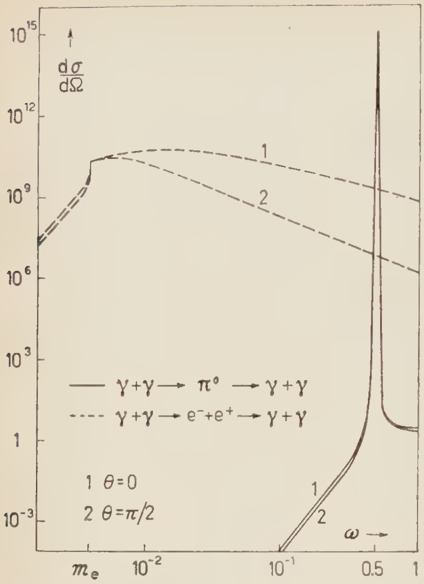

Fig. 1. – Differential cross section for unpolarized photon-photon scattering (cm system). The unit is 10^{-11} cm²/sterad; the unit of energy is $m_\pi c^2$.

The experimental value (³) of $\tau = 2.2 \cdot 10^{-16}$ sec. then gives

$$m_\pi f = 1.5 \cdot 10^{-3}\,.$$

4. – Discussion.

To see the relative importance of $d\sigma_\pi/d\Omega$ and $d\sigma_e/d\Omega$, the values of these terms at $\theta = 0°$ and $90°$ are plotted in Fig. 1. In the low energy limit ($\omega \ll m_e$) both terms are seen to tend to zero as ω^6 and the ratio is

$$\frac{d\sigma_\pi}{d\Omega}\Big/\frac{d\sigma_e}{d\Omega} \sim 10^{-7}\left(\frac{m_e}{m_\pi}\right)^6 \simeq 10^{-22}\,. \tag{15}$$

showing dominant contribution from electron-positron pair states. In the neighborhood of $\omega = m_\pi/2$, however, there is a sharp resonance of $d\sigma_\pi/d\Omega$ whose maximum value is

$$\frac{d\sigma_\pi}{d\Omega}\left(\omega = \frac{m_\pi}{2}\right) = \left(\frac{f^2}{2^4(2\pi)}\right)^2 \left(\frac{m_\pi}{\Gamma}\right)^2 \simeq 1.15 \cdot 10^{-26}\ \text{cm}^2/\text{sr}\,,$$

and whose width at half-maximum is $1/2\tau$. These numbers that determine the shape of the resonance are related to the imaginary part introduced in the denominators of the pole terms, and the sharpness of the resonance reflects the fact that the lifetime of π^0 is relatively long in nuclear time scale. It may be compared with the maxima of $d\sigma_e/d\Omega$ which were given by KARPLUS and NEUMAN (²) to be $4.1 \cdot 10^{-31}$ cm²/sr for $\theta = 0°$ at 1.75 MeV and $2.8 \cdot 10^{-31}$ cm²/sr for $\theta = 90°$ at 0.7 MeV.

* * *

We are greatly indebted to Professor E. C. G. SUDARSHAN for useful discussions.

(³) Proceedings of the Tenth Annual Rochester Conference. (New York, 1960).

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RIASSUNTO (*)

Si mostra che il contributo degli stati intermedi del pione neutro è altrettanto importante per lo scattering fotone-fotone di bassa energia che gli stati della coppia elettrone-positrone. Per lo scattering non polarizzato si mostra che l'interferenza fra due stati intermedi si annulla e che il contributo degli stati del pione neutro alla sezione d'urto differenziale ha un massimo netto di $1.15\cdot10^{-26}\,\text{cm}^2\,\text{sr}$ per una energia totale dei fotoni corrispondente alla massa del pione. Si confrontano le sezioni d'urto in avanti ed ortogonali per due stati intermedi.

(*) *Traduzione a cura della Redazione.*

Diffraction Scattering of Elementary Particles.

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(ricevuto il 21 Aprile 1961)

Summary. — The diffraction scattering of elementary particles is discussed. It is pointed out that there is a possibility of charge exchange diffraction scattering, except when it is inhibited at very high energy by the Pomerančuk theorem (1). The notion of inelastic diffraction scattering, recently discussed by Good and Walker is reformulated in relation to the field theory of unstable particles.

1. – Introduction.

The following remarks on the diffraction scattering of elementary particles have been provoked by experiments on elastic scattering currently being performed at CERN. We shall ignore below all consideration of spin, since they are not essential to our argument.

The quantum theory of diffraction scattering was first given by Bethe and Placzek. To emphasize that the argument depends essentially only on the unitary condition, and can thus be applied directly to elementary particle scattering, we reproduce it in a slightly modified form in the next section. We use covariant normalization to ensure that no energy-dependence is hidden as mass factors in the usual non-relativistic treatment.

In Section 2 we discuss the possibility of charge exchange (or semi-elastic) diffraction scattering, and its relation to theorems of the Pomeranchuk type.

In Section 3 the possibility of inelastic diffraction scattering suggested by Good and Walker (2) is directly related to the theory of unstable particles recently proposed by ourselves (3).

(1) I. Pomerančuk: *J.E.T.P.* (*U.S.S.R.*), **34**, 725 (1958).
(2) M. L. Good and W. D. Walker: *Phys. Rev.*, **120**, 1857 (1960).
(3) P. T. Matthews and A. Salam: *Phys. Rev.*, **112**, 283 (1958); **115**, 1079 (1959).

2. – Diffraction scattering.

The scattering matrix S is related to the T-matrix by

$$S_{if} = \delta_{if} + (2\pi)^4\, i\, \delta(P_i - P_f) T_{if}\,, \tag{2.1}$$

where P is the total energy-momentum vector of the system, and the suffices i and f denote the initial and final states, respectively. The unitarity condition on S implies

$$2\,\mathrm{Im}\, T = T(2\pi)^4\, \delta(P_i - P_f) T^+\,. \tag{2.2}$$

Consider the barycentric system at energy E. Let α denote a two particle channel and θ the scattering angle (spin zero particles). Then

$$2\,\mathrm{Im}\,\langle\alpha, \theta|T(E)|\alpha, 0\rangle = \sum \langle\alpha, \theta|\alpha, l\rangle\cdot$$
$$\cdot\langle\alpha, l|T(2\pi)^4\delta(P_i - P_f)T^+|l, \alpha\rangle\langle\alpha, l|\alpha, 0\rangle\,, \tag{2.3}$$

where $|l, \alpha\rangle$ denotes a particular angular momentum state for the α-channel and

$$\langle\alpha, \theta|\beta, l\rangle = \delta_{\alpha\beta}\sqrt{2l+1}\,P_l(\cos\theta)\,. \tag{2.4}$$

For large E, due to the increase in phase-space for many particle states, the amplitude T is almost purely imaginary (4). Thus, approximately,

$$T(E) \simeq \mathrm{Im}\, T(E)\,. \tag{2.5}$$

Also

$$\langle l, \alpha|T(2\pi)^4\, \delta(P_i - P_f)T^+|\alpha, l\rangle = \sigma_l^{\mathrm{Tot}}(E) 4pE/(2l+1)\,, \tag{2.6}$$

where p is the relative momentum in the c.m. system of the α-channel. Now

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}\Omega} = \left(\frac{1}{8\pi E}\right)^2 |\langle\alpha, \theta|T|\alpha, 0\rangle|^2. \tag{2.7}$$

Thus the differential cross-section for elastic scattering is

$$\frac{\mathrm{d}\sigma_{\mathrm{el}}}{\mathrm{d}\Omega} = \left(\frac{1}{4\pi}\right)^2 |\sum_l p\sigma_l^{\mathrm{Tot}}(E) P_l(\cos\theta)|^2. \tag{2.8}$$

(4) H. Lehmann: (to be published).

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From geometrical considerations we expect

$$\sigma_l^{\text{Tot}}(E) = \left(\frac{2\pi}{p^2}\right)(2l+1)\,a_l(E)\,, \tag{2.9}$$

where

$$a_l(E) \leqslant 1\,,$$

and $a_l(E)$ depends only weakly on both E and l. Substituting (2.9) into (2.8) we obtain

$$\frac{\mathrm{d}\sigma(E)}{\mathrm{d}\Omega} = \frac{1}{p^2}\,\Big|\sum_l (l+\tfrac{1}{2})a_l P_l(\cos\theta)\Big|^2. \tag{2.10}$$

This formula, as it stands, is trivial. The crucial equation is (2.9), which relates the elastic amplitude, a_l, to the total cross-section, through the unitarity condition.

If in (2.10) we replace the sum over l by an integral over impact parameter, b, and use the relation, valid for small θ,

$$(l+\tfrac{1}{2}) \to pb\,, \qquad P_l(\cos\theta) \simeq J_0(pb\sin\theta)\,, \tag{2.11}$$

$$\frac{\mathrm{d}\sigma_{\text{el}}}{\mathrm{d}\Omega} = p^2\left|\int a_b\, J_0(pb\sin\theta)\, b\,\mathrm{d}b\right|^2. \tag{2.12}$$

This is the well-known diffraction formula. For the « black sphere » approximation

$$a_b = 1\,, \qquad b \leqslant R\,,$$

$$a_b = 0\,, \qquad b > R\,,$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{R\cdot J_1\,(pR\sin\theta)}{\sin\theta}\right|^2,$$

giving the standard diffraction peak, with first minimum at

$$\theta = \frac{\pi}{4pR}\,. \tag{2.13}$$

This relationship between the width of the diffraction peak and the radius of interaction is approximately true for any particular assumption about a_b. In terms of the covariant momentum transfer the main diffraction peak is found for values of Δ^2 such that $\Delta^2 \leqslant \Delta_d^2$, where Δ_d is directly related to the least mass, which can be exchanged by the particles.

From (2.10)

$$\sigma_{el}^{l} = \frac{\pi}{p^2}(2l+1)a_l^2 . \tag{2.14}$$

Thus

$$\sigma_{el}^{l}/\sigma_{Tot}^{l} = a^l/2 . \tag{2.15}$$

Taking an average value of the significant a_l $(l < l_{max})$ to be a, where

$$pR = l_{max} ,$$

then

$$\sum^{l_{max}} (2l+1) = p^2R^2 .$$

Thus [5]

$$\sigma_{el}/\sigma_{Tot} = a/2 ,$$

and

$$R^2 = \frac{\sigma_{Tot}^2}{4\pi\sigma_{el}} .$$

The factor a_l as it appears in (2.10) can be written in terms of a complete phase shift

$$\delta = \alpha + i\beta , \qquad \beta > 0, \tag{2.16}$$

as

$$a_l = \frac{1 - \exp[2i\delta]}{i} . \tag{2.17}$$

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[5] For example at 24 GeV/c (lab) p-p collision

$$\sigma_{el} \simeq 7 \text{ mb} .$$

$$\sigma_{Tot} \simeq 42 \text{ mb} .$$

giving

$$a = \tfrac{1}{3} ,$$

$$R \simeq 1.4 \cdot 10^{-13} \text{ cm} .$$

For π^--p collisions at 16 GeV/c (lab)

$$\sigma_{el} \simeq 4.5 \text{ mb} ,$$

$$\sigma_{Tot} \simeq 25 \text{ mb} ,$$

giving

$$a \simeq \tfrac{3}{8}$$

$$R \simeq 1.0 \cdot 10^{-13} \text{ cm} .$$

(These data are preliminary results privately communicated by Dr. B. FRENCH).

The « black sphere » approximation is to take β large ($e^{-\beta} \ll 1$) for those values of l with impact parameters corresponding to hitting the target. Experimentally, however, the elementary particles are very far from being black spheres ([5]), and the appropriate approximation is (2.5), which is satisfied if α is small.

3. – Semi-elastic diffraction scattering.

Consider the scattering of systems (such as p-n or π-p) which are mixtures of isotopic spin states. Then we may have either elastic scattering or charge exchange scattering, which we shall refer to as semi-elastic. Making an isotopic spin analysis, for these particular initial and final states,

$$\left\{\begin{aligned} \langle f|T|i\rangle &= \sum_I \langle f|I\rangle T_I \langle I|i\rangle \\ &\equiv \sum_I \alpha_I^{fi}\, T_I\,, \end{aligned}\right. \tag{3.1}$$

where

$$\sum_I \alpha_I^{fi} = \delta_{fi} \tag{3.2}$$

Then (2.10) is replaced by

$$\frac{\mathrm{d}\sigma_{fi}}{\mathrm{d}\Omega} = \frac{1}{p^2}\Big|\sum_l (l+\tfrac{1}{2}) \sum_I \alpha_I^{fi} a_I^l P_l(\cos\theta)\Big|^2 . \tag{3.3}$$

If a_I^l is independent of I, that is to say, if the total cross-section is the same in all isotopic spin states, this term can be taken outside the summation In this case, by (3.2), the semi-elastic diffraction scattering vanishes identically. However, in general, there is no reason for this to be the case and one would expect a sizeable amount of semielastic diffraction.

This implies, for example, an appreciable amount of charge exchange, (ce), diffraction scattering in p-n collisions. Assuming one can average similarly over l-values in the two i-spin states the diffraction cross-section in terms of the i-spin amplitudes are in the ratio

$$\frac{\sigma_{\text{ce}}}{\sigma_{\text{el}}} = \frac{|a_0 - a_1|^2}{|a_0 + a_1|^2}.$$

In the « black sphere » limit, ($a_0 \simeq a_1 \simeq 1$), the ratio is vanishingly small, but as mentioned above, this is found experimentally not to be the case ([5]). The same formula also applies to K⁻-p collisions, for which the charge exchange

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diffraction might again be appreciable. In both cases the mechanism could be important for producing neutral beams.

An exception to this argument is the case of π^--nucleon scattering, where the Pomeranchuk theorem [1] implies that

$$\sigma^{\text{Tot}}_{\frac{3}{2}} = \sigma^{\text{Tot}}_{\frac{1}{2}}$$

at high energies. From the work of VON DARDEL [7], it appears that this limit is reached at about 3 GeV energy (c.m.). An observation of the energy dependence of charge exchange scattering would provide an interesting check on these results.

Another exception is the case of semi-elastic diffraction scattering of θ^0_2 (where the mixture is of strangeness rather than I-spin). Here again a Pomerančuk theorem relating $\sigma_{K\mathcal{N}}$ and $\sigma_{\bar{K}\mathcal{N}}$ implies that in the Pomerančuk energy limit the effect observed by PICCIONI [8] and collaborators should disappear.

4. - Inelastic diffraction scattering.

Finally we would like to reformulate an idea put forward by GOOD and WALKER [2]; and relate it to the theory of unstable particles recently proposed by us [3].

Let the complete renormalized Feynman propagator of the diffracted particle be

$$\Delta_c(x) = i\langle T(\varphi(x), \varphi(0))\rangle_0 = (2\pi)^{-4}\int_0^\infty \mathrm{d}\varkappa^2 \int \frac{\varrho(\varkappa^2)\exp[ipx]}{p^2 - \varkappa^2 + i\varepsilon}\,\mathrm{d}^4p \tag{4.1}$$

$$= (2\pi)^{-4}\int \Delta_c(p^2)\exp[ipx]\,\mathrm{d}^4p\,.$$

The Källén-Lehmann [10] spectral function is

$$\varrho(p^2) = \sum_\alpha |\langle 0|\varphi|p, \alpha\rangle|^2\,, \tag{4.2}$$

([6]) G. COCCONI, A. N. DIDDENS, E. LILLETHUN and A. M. WETHERELL: *Phys. Rev. Lett.*, **6**, 231 (1961).

([7]) G. VON DARDEL, D. H. FRISCH, R. MERMOD, R. H. MILBURN, P. A. PIROUÉ, M. VIVARGENT, G. WEBER and K. WINTER: *Phys. Rev. Lett.*, **5**, 333 (1960).

([8]) F. MULLER, R. W. BIRGE, W. B. FOWLER, R. H. GOOD, W. HIRSCH, R. P. MATSEN, L. OSWALD, W. M. POWELL, H. S. WHITE and O. PICCIONI: *Phys. Rev. Lett.*, **4**, 418 (1960).

([9]) G. KÄLLÉN: *Handb. d. Phys.*, **5**, No. 1 (1958).

([10]) H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954).

where the summation is over all states congruent to the diffracted particle with total four-momentum p. We define

$$\varrho_i(p^2) = |\langle 0|\varphi|p, i\rangle|^2 ,$$

where $|p, i\rangle$ is a particular type of state with the nature of the particles specified. If the diffraction scattering of the particle mass m, off a particle, M, at c.m. energy U, and covariant momentum transfer Δ^2, is given by

$$(4.3) \qquad \mathrm{d}\sigma_{\mathrm{el}} = \frac{1}{\text{flux}} |F(U^2, \Delta, \omega^2)|^2 \, \delta(\omega^2 - n^2)\, \delta(\omega'^2 - M^2) \cdot \\ \cdot (2\pi)^4 \delta(P_i - P_f) \frac{\mathrm{d}^4\omega}{(2\pi)^3} \frac{\mathrm{d}^4\omega'}{(2\pi)^3} ;$$

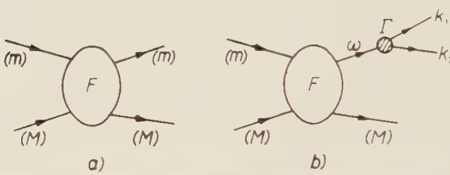

Fig. 1. - If (a) is the matrix element for diffraction scattering, eq. (4.3), then (b) is the matrix element for inelastic diffraction giving rise to eq. (4.4).

then the cross-section for inelastic diffraction, with the appearance of particles i in the diffraction peak, is, (see Fig. 1),

$$(4.4) \qquad \mathrm{d}\sigma_i = \frac{1}{\text{flux}} |F(U^2, \Delta^2, \omega^2)\Delta_c(\omega^2)\Gamma_i^2(\omega, k_1, k_2)|^2 \cdot \\ \cdot \frac{1}{(2\pi)^3} \delta(\omega - k_1 - k_2)\Delta^+(k_1)\Delta^+(k_2)\,\mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\delta(\omega^2 - p^2)\,\mathrm{d}p^2 \cdot \\ \cdot \frac{\mathrm{d}^4\omega}{(2\pi)^3} \delta(\omega'^2 - M^2)(2\pi)^4\delta(P_i - P_f)\frac{\mathrm{d}^4\omega'}{(2\pi)^3} ,$$

where

$$\Delta^+(k) = \theta(k_0)\,\delta(k^2 - m^2) ,$$

and

$$\Gamma_i(\omega, k_1, k_2)$$

is the vertex part corresponding to the « decay » of the diffracted particle (four-vector ω) into the state (i), (two particles k_1, k_2). We have ignored the possibility of final state interaction between the i-particles and the rest of the system. Now it has been shown by us ([3]) in connection with the decay

of unstable particles, that this is just

$$\text{(4.5)}\qquad \mathrm{d}\sigma_i = \frac{1}{\text{flux}} F(U^2, \Delta^2, \omega^2)\,\varrho_i(\omega^2)\,\delta(\omega^2 - p^2)\,\delta(\omega'^2 - M^2)\cdot \cdot(2\pi)^4\,\delta(P_i - P_f)\,\frac{\mathrm{d}^4\omega}{(2\pi)^3}\,\frac{\mathrm{d}^4\omega'}{(2\pi)^3}\,\mathrm{d}p^2\,.$$

Summing over i, we have, for the total inelastic diffraction σ_{in},

$$\text{(4.6)}\qquad \therefore \frac{\mathrm{d}\sigma_{\text{el}}}{\mathrm{d}\Delta^2}\bigg/\frac{\mathrm{d}\sigma_{\text{in}}}{\mathrm{d}\Delta^2\,\mathrm{d}p^2} \simeq \frac{|F(U^2, \Delta^2, m^2)|^2}{|F(U^2, \Delta^2, p^2)\,\varrho(p^2)|^2}\,.$$

Since it may be assumed that F does not depend critically on p^2, we see that an investigation of inelastic diffraction scattering amounts to a rather direct observation of the spectral function in the continuum region.

The factor F is appreciable for values of

$$\text{(4.7)}\qquad \Delta^2 < \Delta_d^2\,.$$

The maximum value of p^2 in this region is determined by the relation

$$\text{(4.8)}\qquad \Delta_d^2 U^2 \left[1 - \frac{p^2 + m^2 + 2M^2 + \Delta_d^2}{U^2} + \frac{(p^2 - M^2)(m^2 - M^2)}{U^4}\right] = = \frac{(p^2 + m^2 - 2M^2)(p^2 - m^2)\,M^2}{U^2}\,.$$

Thus for large U^2, in nucleon-nucleon collisions for example,

$$\text{(4.9)}\qquad p^2 = N^2 + \frac{\Delta_d U^2}{N}\,,$$

where N is the nucleon mass and Δ_d is of the order of the pion mass. This sets the upper limit on the value of p^2 for which one may expect inelastic diffraction scattering.

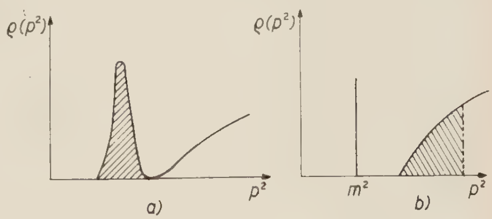

Fig. 2. – Schematic plots of $\varrho(p^2)$ for, (a) unstable and (b) stable particles. The shaded areas determine, for (a) the characteristics of decay, and for (b) the inelastic diffraction.

Schematically what is happening is rather clear. The ϱ factors of stable and unstable particles are as shown in Fig. 2. By studying the mean mass and mass spread (life-time) of the decay products of unstable particles we investigate $\varrho(p^2)$ for values of p^2 in the neighbourhood of the mean mass. In inelastic diffraction a stable particle is produced in a mass state corresponding

to the continuum and one can, in a closely analogous manner study the spectral function $\varrho(p^2)$ in the lower regions of the continuum, the upper limit on p^2 being set by (4.8).

RIASSUNTO (*)

Discutiamo lo scattering di diffrazione delle particelle elementari. Facciamo notare che c'è la possibilità di uno scattering di diffrazione con scambio di carica, salvo quando esso sia proibito ad altissime energie dal teorema di Pomerančuk. In relazione alla teoria di campo delle particelle instabili, riformuliamo la nozione di scattering di diffrazione anelastica, discusso recentemente da GOOD e WALKER.

(*) *Traduzione a cura della Redazione.*

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On the $\mu \to 3e$ Decay as a Test of the Applicability of the Perturbation Expansion in the Theory of Weak Interactions.

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Summary. — As a test of the applicability of the perturbation expansion in the theory of weak interactions the $\mu \to 3e$ decay has been considered. The weak interactions are given in the usual current $\times$ current form including quadratic terms like $(\bar{e}\nu)(\bar{\nu}e)$. The best value so far available for the branching ratio $\mu \to 3e/\mu \to e+\nu+\bar{\nu}$ then requires a cut-off at 90 GeV to make the theoretical predictions compatible with the experimental findings. This cut-off is in agreement with the cut-off obtained in $\mu \to e+\gamma$ decay discussed earlier by Ioffe.

1. – Introduction.

Several authors (1) have lately raised the question of whether the perturbation expansion in the theory of weak interactions is legitimate or not. As the coupling constant for weak interactions is not dimensionless the weak interactions grow with energy, as was first noted by Heisenberg, to become strong at an energy of the order of magnitude of 10^3 GeV, unless somehow a cut-off is acting before this energy range is reached. This is so because the dimensional character of the coupling constant leads to a cut-off-dependent expansion parameter in the usual perturbation expansion. Therefore, if no cut-off is present the contributions from higher order terms in the expansion become essential and results obtained from lowest order calculations are not reliable. On the other hand, if there is an effective cut-off below the critical

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(1) B. L. Ioffe: *Proc. of the 1960 Ann. Int. Conference on High Energy Physics at Rochester*, p. 561; L. B. Okun: *Proc. of the 1960 Ann. Int. Conference on High Energy Physics at Rochester*, p. 743.

energy range, we face a situation within the theory of weak interactions similar to what has been previously found in the theory of strong interactions, that is, the interactions have some kind of form factors although in the weak interactions these form factors become important at such a low energy that the weak interactions have not yet become strong. In view of the near equality of the renormalized coupling constants in β- and μ-decay it is generally believed that the latter alternative is correct.

A first step towards better understanding of these form factors is to investigate at what energy the interactions are effectively cut off. In order to avoid the complications from strong interactions it is most appealing to deal with processes where only weak and electromagnetic interactions are present. In this way one has been led to investigate the different decay modes of the μ-meson. The different modes that can contribute to the μ-meson decay are obviously depending on the type of weak interaction Hamiltonian one adopts. There are mainly four different schemes presently discussed (1). We shall here adopt the scheme where the lepton interaction is given by the following Hamiltonian

$$H_W = \frac{G}{\sqrt{2}} J_\alpha^+ J^\alpha , \tag{1.1}$$

where G denotes the unrenormalized weak interaction coupling constant and the current J_α is defined by

$$J_\alpha = \overline{\psi}_e \gamma_\alpha (1 + \gamma_5) \psi_\nu + \overline{\psi}_\mu \gamma_\alpha (1 + \gamma_5) \psi_\nu . \tag{1.2}$$

It should be noted that within this scheme quadratic terms of the type $(\bar{e}\nu)(\bar{\nu}e)$ are included and only one kind of neutrinos is allowed for. This scheme is in excellent agreement with the experimental findings and has many appealing features as compared to other possible schemes.

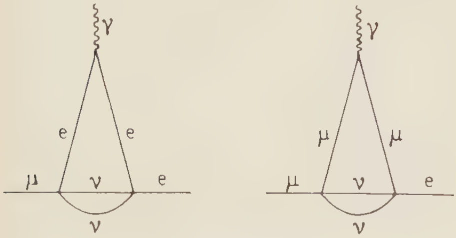

Fig. 1. – Feynman diagrams for the $\mu \to e + \gamma$ decay.

The cut-off value is further depending on how the cut-off is introduced in theory. We shall here apply the Feynman regularization of the propagators (2).

The first μ-meson decay investigated along these lines is the $\mu \to$ $\to e + \gamma$ which can occur according to the Feynman graphs of Fig. 1 within the scheme adopted here.

(2) S. S. Schweber, H. A. Bethe and F. de Hoffmann: *Mesons and Fields*, vol. I (Evanston, Ill., 1956).

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From these diagrams IOFFE (3) obtains the branching ratio

$$\frac{W_{e+\gamma}}{W_{e+\nu+\bar{\nu}}} = \frac{2}{3\pi^5}\, e^2 G^2 \Lambda^4 \left[\ln \frac{\Lambda^2}{m_\mu^2}\right]^2 . \tag{1.3}$$

The experimental upper limit for this branching ratio is given by $2 \cdot 10^{-6}$ from which follows as an upper limit of the cut-off Λ

$$\Lambda \leqslant 50 \text{ GeV} .$$

We shall in this paper investigate the $\mu \to 3e$ decay, which in the adopted scheme can proceed through the following graph

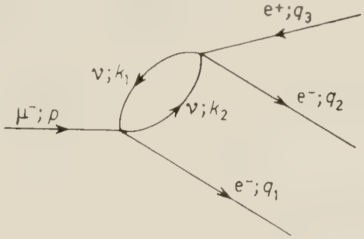

Fig. 2. – Feynman diagram for the $\mu \to 3e$ decay.

Throughout the calculations we shall use notations in accordance with reference (2). In a forthcoming paper the analysis will be extended to other processes which can possibly give information with respect to the cut-off in the theory of weak interactions.

2. – The S-matrix element for the $\mu \to 3e$ decay.

The appropriate part of the weak interaction Hamiltonian for the process studied here is given by

$$H_W = \frac{G}{\sqrt{2}} \left[\bar{\psi}_e \Gamma^\varkappa \psi_\nu \bar{\psi}_\nu \Gamma_\varkappa \psi_\mu + \bar{\psi}_e \Gamma^\varkappa \psi_\nu \bar{\psi}_\nu \Gamma_\varkappa \psi_e\right] , \tag{2.1}$$

where we have introduced the abbreviated notation $\Gamma_\varkappa = \gamma_\varkappa(1+\gamma_5)$. It is convenient for explicit calculations to rewrite this Hamiltonian by means of a

(3) B. L. IOFFE: *Žurn. Èksp. Teor. Fiz.*, **38**, 1608 (1960).

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Fierz transformation in the following way

$$H_W = \frac{G}{\sqrt{2}} [\bar{\psi}_e \Gamma^\varkappa \psi_\mu \bar{\psi}_\nu \Gamma_\varkappa \psi_\nu + \bar{\psi}_e \Gamma^\varkappa \psi_e \bar{\psi}_\nu \Gamma_\varkappa \psi_\nu] \,. \tag{2.2}$$

We obtain the S-matrix element from this Hamiltonian by standard methods. To avoid the quadratic divergences which appear we introduce a cut-off in the theory by a Feynman regularization of the neutrino propagators, that is by the following replacement

$$\frac{\mathbf{k}}{k^2} \to -\int_0^{\Lambda^2} \mathrm{d}\lambda \frac{\mathbf{k}}{[k^2-\lambda]^2} \,. \qquad \boldsymbol{k} - \gamma^\mu k_\mu \,.$$

With the notations from Fig. 2 we then obtain after integrating over all dummy variables except k_2

$$\begin{aligned} S_{fi} = &-\frac{G^2}{4}\cdot\frac{1}{(2\pi)^6}\cdot\delta^4(p-\sum_{i=1}^3 q_i)\left[\frac{m_e^3 m_\mu}{E(q_1)E(q_2)E(q_3)E(p)}\right]^{\frac{1}{2}}\cdot \\ &\cdot\frac{1}{\sqrt{2}}\left\{\int_0^{\Lambda^2}\mathrm{d}\lambda_1\int_0^{\Lambda^2}\mathrm{d}\lambda_2\int\mathrm{d}^4k\bar{u}^r(q_1)\Gamma^\varkappa u^s(p)\cdot\bar{u}^t(q_2)\Gamma^\lambda v^u(q_3)\cdot\right. \\ &\left.\cdot\mathrm{Tr}\left[\frac{\mathbf{k}}{[k^2-\lambda_1]^2}\Gamma_\varkappa\frac{\mathbf{k}+\mathbf{q}_1-\mathbf{p}}{[k^2+2k(q_1-p)+(q_1-p)^2-\lambda_2]^2}\Gamma_\lambda\right]-[q_1\leftrightarrow q_2]\right\}. \end{aligned} \tag{2.3}$$

The S-matrix element has been antisymmetrized in the final state electrons as required by the Pauli principle ([4]). Noting the following properties for the helicity projection operators

$$\tfrac{1}{2}(1\pm\gamma_5)^2 = \tfrac{1}{2}(1\pm\gamma_5)\,; \qquad \tfrac{1}{4}(1+\gamma_5)(1-\gamma_5) = 0$$

and that the γ_5-matrix anticommutes with the other γ-matrices we can write (2.3)

$$\begin{aligned} S_{fi} = &-\frac{G^2}{2\sqrt{2}}\cdot\frac{1}{(2\pi)^6}\,\delta^4(p-\sum_{i=1}^3 q_i)\left[\frac{m_e^3 m_\mu}{E(q_1)E(q_2)E(q_3)E(p)}\right]^{\frac{1}{2}}\bar{u}^r(q_1)\Gamma^\varkappa u^s(p)\cdot \\ &\cdot\mathrm{Tr}\,[\gamma_\alpha\Gamma_\varkappa\gamma_\beta\gamma_\lambda]\bar{u}^t(q_2)\Gamma^\lambda v^u(q_3)\left\{\int_0^{\Lambda^2}\mathrm{d}\lambda_1\int_0^{\Lambda^2}\mathrm{d}\lambda_2\int\mathrm{d}^4k\,\frac{k^\alpha}{[k^2-\lambda_1]^2}\cdot\right. \\ &\left.\cdot\frac{k^\beta+q_1^\beta-p^\beta}{[k^2+2k(q_1-p)+(q_1-p)^2-\lambda_2]^2}-[q_1\leftrightarrow q_2]\right\}. \end{aligned} \tag{2.4}$$

([4]) N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields* (New York, 1959), p. 259.

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We shall first carry out the integrations and denote the integral in (2.4) by

$$I = \int_0^{\Lambda^2} \mathrm{d}\lambda_1 \int_0^{\Lambda^2} \mathrm{d}\lambda_2 \int \mathrm{d}^4 k \, \frac{k^\alpha}{[k^2 - \lambda_1]^2} \cdot \frac{k^\beta + q_1^\beta - p^\beta}{[k^2 + 2k(q_1 - p) + (q_1 - p)^2 - \lambda_2]^2} . \tag{2.5}$$

The k-integration is most easily performed after a Feynman parametrization of (2.5), that is by observing the following identity

$$\frac{1}{a^2 b^2} = \int_0^1 \frac{6z(1-z)\,\mathrm{d}z}{[az + b(1-z)]^4} ,$$

where possible singularities are avoided by a detour in the complex z-plane. In this way we obtain

$$I = \int_0^{\Lambda^2} \mathrm{d}\lambda_1 \int_0^{\Lambda^2} \mathrm{d}\lambda_2 \int_0^1 \mathrm{d}z \cdot 6z(1-z) \cdot \tag{2.6}$$

$$\cdot \int \mathrm{d}^4 k \, \frac{k^\alpha (k^\beta + q_1^\beta - p^\beta)}{\{[k + (q_1 - p)z]^2 + (q_1 - p)^2 z(1-z) + (\lambda_1 - \lambda_2) z - \lambda_1\}^4} .$$

Introducing the new variable $k' = k + (q_1 - p)z$ and exploiting the symmetry properties of the integrand we arrive at

$$I = \frac{i\pi^2}{2} \int_0^{\Lambda^2} \mathrm{d}\lambda_1 \int_0^{\Lambda^2} \mathrm{d}\lambda_2 \int_0^1 \mathrm{d}z \, z(z-1) \cdot \tag{2.7}$$

$$\cdot \left\{ \frac{g^{\alpha\beta}}{\lambda_1 - (\lambda_1 - \lambda_2)z - (q_1 - p)^2 z(1-z)} - \frac{2z(1-z)(q_1^\alpha - p^\alpha)(q_1^\beta - p^\beta)}{[\lambda_1 - (\lambda_1 - \lambda_2)z - (q_1 - p)^2 z(1-z)]^2} \right\} .$$

Neglecting the second term as it does not contribute to the highest order term in the cut-off we remain with

$$I = \frac{i\pi^2}{2} g^{\alpha\beta} \int_0^1 \mathrm{d}z \, z \left\{ \Lambda^2 \ln \frac{\Lambda^2 z - (q_1 - p)^2 z(1-z)}{\Lambda^2 - (q_1 - p)^2 z(1-z)} - \right. \tag{2.8}$$

$$- (q_1 - p)^2 (1-z) \ln \frac{\Lambda^2 z - (q_1 - p)^2 z(1-z)}{(q_1 - p)^2 z(1-z)} +$$

$$+ \frac{\Lambda^2 (1-z) - (q_1 - p)^2 z(1-z)}{z} \ln \frac{\Lambda^2 (1-z) - (q_1 - p)^2 z(1-z)}{\Lambda^2 - (q_1 - p)^2 z(1-z)} ,$$

after the λ-integrations have been carried out. If we neglect terms of the order $(q_1-p)^2/\Lambda^2$ we get

$$I = -\frac{i\pi^2}{4}\Lambda^2 g^{\alpha\beta}. \tag{2.9}$$

Inserting this in (2.4) we finally obtain

$$S_{fi} = -\frac{i\pi^2 G^2}{8\sqrt{2}}\cdot\frac{1}{(2\pi)^6}\,\delta^4(p-\sum_{i=1}^{3}q_i)\left[\frac{m_e^3 m_\mu}{E(q_1)E(q_2)E(q_3)E(p)}\right]^{\frac{1}{2}}\cdot \\ \cdot\{g^{\alpha\beta}\Lambda^2\cdot[M(q_1,q_2)-M(q_2,q_1)]\}, \tag{2.10}$$

with

$$M(q_1,q_2) = \overline{u}^r(q_1)\Gamma^\varkappa u^s(p)\,\mathrm{Tr}\,[\gamma_\alpha\Gamma_\varkappa\gamma_\beta\gamma_\lambda]\,\overline{u}^t(q_2)\,\Gamma^\lambda v^u(q_3). \tag{2.11}$$

3. – Calculation of the transition probability.

If we define a matrix $\mathscr{M}$ by

$$S = \delta^4(P_f - P_i)\mathscr{M},$$

with P_f and P_i denoting the four-momentum of the final state and the initial state respectively, then the transition probability per unit time can be written

$$\Gamma = \frac{1}{2\pi}\sum|\langle f|\mathscr{M}|i\rangle|^2\delta^4(P_f-P_i).$$

The summation here stands for integration over the final state three-momenta, averaging over initial spins and summation over final spins. With (2.10) we therefore, obtain

$$\Gamma_{\mu\to3e} = \frac{1}{(2\pi)^{13}}\cdot\frac{\pi^4}{128}\,G^4\Lambda^4\int d^3q_1\int d^3q_2\int d^3q_3\,\delta^4(p-\sum_{i=1}^{3}q_i)\,\frac{m_e^3 m_\mu}{E(q_1)E(q_2)E(q_3)E(p)}\cdot \\ \cdot g^{\alpha\beta}g^{\alpha'\beta'}\cdot\tfrac{1}{2}\sum_{\text{spins}}\{M(q_1,q_2)M^+(q_1,q_2)-M(q_1,q_2)M^+(q_2,q_1)- \\ -M(q_2,q_1)M^+(q_1,q_2)+M(q_2,q_1)M^+(q_2,q_1)\}, \tag{3.1}$$

where

$$\sum_{\text{pins}} M(q_1,q_2)M^+(q_1,q_2) = -\,\mathrm{Tr}\,[\Gamma^\varkappa\Lambda_+(p)\Gamma^\sigma\Lambda_+(q_1)]\,\mathrm{Tr}\,[\Gamma^\lambda\Lambda_-(q_3)\Gamma^\tau\Lambda_+(q_2)]\cdot \\ \cdot\mathrm{Tr}\,[\gamma_\alpha\Gamma_\varkappa\gamma_\beta\gamma_\lambda]\cdot\mathrm{Tr}\,[\gamma_\tau\gamma_{\beta'}\Gamma_\sigma\gamma_{\alpha'}],$$

$$\sum_{\text{spins}} M(q_1, q_2) M^+(q_2, q_1) = -\operatorname{Tr}[\Gamma^\varkappa \Lambda_+(p) \Gamma^\sigma \Lambda_+(q_2) \Gamma^\lambda \Lambda_-(q_3) \Gamma^\tau \Lambda_+(q_1)] \cdot$$
$$\cdot \operatorname{Tr}[\gamma_\alpha \Gamma_\varkappa \gamma_\beta \gamma_\lambda] \cdot \operatorname{Tr}[\gamma_\tau \gamma_{\beta'} \Gamma_\sigma \gamma_{\alpha'}] .$$

For convenience of writing we have denoted the energy projection operators

$$\Lambda_\pm(p) = \frac{m \pm \mathbf{p}}{2m} ,$$

The third and fourth terms of (3.1) are obtained by the replacement $q_1 \leftrightarrow q_2$ in the first two terms. After the trace calculations are done we remain with

$$\Gamma_{\mu\to 3e} = \frac{1}{(2\pi)^9} \left[\frac{G\Lambda}{\sqrt{2}}\right]^4 \int \mathrm{d}^3 q_1 \int \mathrm{d}^3 q_2 \int \mathrm{d}^3 q_3 \, \delta^4(p - \sum_{i=1}^{3} q_i) \cdot \frac{(pq_2)(q_1q_3) + (pq_1)(q_2q_3) + 2(pq_3)(q_1q_2)}{E(q_1)E(q_2)E(q_3)E(p)} . \tag{3.2}$$

This expression holds in any Lorentz frame. We shall evaluate the transition probability per unit time in the muon rest frame, where (3.2) takes the form

$$\Gamma_{\mu\to 3e} = \frac{1}{(2\pi)^9} \cdot \left[\frac{G\Lambda}{\sqrt{2}}\right]^4 \int \mathrm{d}^3 q_1 \int \mathrm{d}^3 q_2 \int \mathrm{d}^3 q_3 \, \delta(m_\mu - \sum_{i=1}^{3} E(q_i)) \, \delta^3(\sum_{i=1}^{3} \boldsymbol{q}_i) \cdot \left\{4 - \frac{2\boldsymbol{q}_1 \cdot \boldsymbol{q}_2}{E(q_1)E(q_2)} - \frac{\boldsymbol{q}_1 \cdot \boldsymbol{q}_3}{E(q_1)E(q_3)} - \frac{\boldsymbol{q}_2 \cdot \boldsymbol{q}_3}{E(q_2)E(q_3)}\right\} , \tag{3.3}$$

or due to the symmetry in $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$

$$\Gamma_{\mu\to 3e} = \frac{4}{(2\pi)^9} \left[\frac{G\Lambda}{\sqrt{2}}\right]^4 \int \mathrm{d}^3 q_1 \int \mathrm{d}^3 q_2 \int \mathrm{d}^3 q_3 \, \delta(m_\mu - \sum_{i=1}^{3} E(q_i)) \cdot \delta^3(\sum_{i=1}^{3} \boldsymbol{q}_i) \left\{1 - \frac{\boldsymbol{q}_1 \cdot \boldsymbol{q}_2}{E(q_1)E(q_2)}\right\} . \tag{3.4}$$

If we neglect terms of the order m_e/m_μ and perform the $\boldsymbol{q}_3$ integration by means of the three-dimensional δ-function we obtain the following result

$$\Gamma_{\mu\to 3e} = \frac{4}{(2\pi)^9} \left[\frac{G\Lambda}{\sqrt{2}}\right]^4 \int \mathrm{d}^3 q_1 \int \mathrm{d}^3 q_2 \, \delta\left[m_\mu - \sum_{i=1}^{3} E(q_i)\right] \{1 - \cos\theta\} , \tag{3.5}$$

where q_i now stands for the length of the vector $\boldsymbol{q}_i$ and further

$$q_3 = \sqrt{(\boldsymbol{q}_1 + \boldsymbol{q}_2)^2} .$$

The angle between $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ has been denoted by θ. Introducing spherical co-ordinates one has

$$(3.6)\qquad \Gamma_{\mu\to 3e} = \frac{8}{(2\pi)^7}\left[\frac{GA}{\sqrt{2}}\right]^4 \int \mathrm{d}q_1\, q_1^2 \int \mathrm{d}q_2\, q_2^2 \int \mathrm{d}(\cos\theta)\,\delta\Big[m_\mu - \sum_{i=1}^{3} E(q_i)\Big]\{1-\cos\theta\}\,.$$

To carry out the q_2-integration we make use of the remaining δ-function. We have in general

$$\int \delta[\varphi(x)]f(x)\,\mathrm{d}x = \frac{1}{|\partial\varphi/\partial x|_{x=x_0}}\, f(x_0)\,,$$

where x_0 is defined by $\varphi(x_0)=0$. In our case we have

$$(3.7)\qquad \varphi(q_2) = m - q_1 - q_2 - \sqrt{q_1^2+q_2^2+2q_1q_2\cos\theta}\,.$$

This function of q_2 is zero for

$$(3.8)\qquad q_2' = \frac{1}{2}\cdot\frac{m_\mu(m_\mu-2q_1)}{m_\mu - q_1(1-\cos\theta)}\,,$$

and we get

$$(3.9)\qquad \left|\frac{\partial\varphi}{\partial q_2}\right|_{q_2=q_2'} = 1 + \frac{\alpha + 2q_1y - 2y(m_\mu-y)}{2y(m_\mu-q_1)-\alpha}\,,$$

where we have introduced

$$y = m_\mu - q_1(1-\cos\theta)\,,$$

$$\alpha = m_\mu(m_\mu - 2q_1)\,.$$

If we introduce y as a new variable of integration we can write (3.6) after the q_2-integration has been carried out

$$(3.10)\qquad \Gamma_{\mu\to 3e} = \frac{1}{(2\pi)^7}\left[\frac{GA}{\sqrt{2}}\right]^4 \int_0^{m_\mu/2}\mathrm{d}q_1 \int_{m_\mu-2q_1}^{m_\mu}\mathrm{d}y\, \frac{\alpha^2[2y(m_\mu-q_1)-\alpha](m_\mu-y)}{y^4}\,,$$

or finally after the remaining integrations are performed

$$(3.11)\qquad \Gamma_{\mu\to 3e} = \frac{1}{24(2\pi)^7}\left[\frac{GA}{\sqrt{2}}\right]^4\cdot m_\mu^5\,.$$

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4. – Comparison with the experimental results.

Experimentally, an upper limit of the branching ratio for the muon decay into three electrons as compared to the normal mode of decay [$\mu \to e+\nu+\bar{\nu}$] has been determined ([5]). The transition probability per unit time for the latter decay is given by ([6])

$$\Gamma_{\mu\to e+\nu+\bar{\nu}} = \frac{G^2 m_\mu^5}{24(2\pi)^3}\,. \tag{4.1}$$

From (3.11) and (4.1) we obtain the following theoretical value for the branching ratio

$$B_{\text{th}} = \frac{3G\Lambda^4}{256\pi^4}\,, \tag{4.2}$$

or inserting

$$G \simeq \frac{10^{-5}}{m_p^2}\,; \qquad m_p = \text{proton mass}\,,$$

$$B_{\text{th}} = \frac{10^{-10}}{64\pi^4}\left[\frac{\Lambda}{m_p}\right]^4. \tag{4.3}$$

With the upper limit for this branching ratio experimentally determined to be $B_{\text{exp}} \leqslant 10^{-6}$ we obtain for the cut-off

$$\Lambda \leqslant 90 \text{ GeV}\,.$$

5. – Conclusion.

The upper limit for the cut-off obtained in this study of the muon decay into three electrons is in agreement with the result for the cut-off entering the $\mu \to e+\gamma$ as calculated by Ioffe ([1]). Earlier calculations of the $\mu \to 3e$ decay by Ioffe gave considerably higher values for the cut-off. That is due to the fact that Ioffe considered more complicated diagrams in order to avoid the self-coupling of the weak interaction currents. There is, however, no contra-

([5]) R. R. Crittenden and W. D. Walker: preprint.
([6]) E. J. Konopinski: *Ann. Rev. Nucl. Sci.*, **9**, 148 (1959).

diction so far to the scheme including these self-couplings, although the experimental material regarding the existence or non-existence of them is admittedly very meagre.

* * *

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RIASSUNTO (*)

Come controllo dell'applicabilità della teoria delle perturbazioni alla teoria delle interazioni deboli, si è preso in considerazione il decadimento $\mu \to 3e$. Le interazioni deboli sono espresse nella forma usuale corrente $\times$ corrente, compresi termini quadratici come $(\bar{e}\nu)(\bar{\nu}e)$. Il migliore valore, sinora disponibile, per il rapporto di branching $\mu \to 3e/\mu \to e+\nu+\bar{\nu}$, richiede quindi un cut-off a 90 GeV per rendere le predizioni teoriche compatibli con le risultanze sperimentali. Questo cut-off è in accordo con il cut-off ottenuto nel decadimento $\mu \to e+d$, precedentemente discusso da Ioffe.

(*) *Traduzione a cura della Redazione.*

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A Time-Dependent Approach to Rearrangement Collisions (*).

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Summary. — A time-dependent theory of rearrangement collisions is presented which forms a simple generalization of the well-known scattering theory. The S matrix is obtained for such processes and it is found that the matrix elements appear in forms similar to those of scattering theory. As an illustration, some elementary applications are discussed.

1. – Introduction.

The S-matrix theory of scattering [1-3] is concerned with the transition from an initial state of a system of particles to a final state of a system consisting of particles of the *same type* as in the initial system. This is a particular case, though an important one, of a more general class of phenomena. When two systems of particles (nuclei, for example) collide, they often give rise, after collision, to systems which may comprise nuclei different from those of the initial system. Such collisions are referred to as rearrangement collisions or collisions leading to reaction channels. It is the purpose of this paper to suggest a simple generalization of the usual S-matrix theory to include rearrangement collisions.

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(**) Junior Research Fellow of the Council of Scientific and Industrial Research (India).

(1) B. A. Lippmann and J. Schwinger: *Phys. Rev.*, **79**, 469 (1950).

(2) M. Gellmann and M. L. Goldberger: *Phys. Rev.*, **91**, 398 (1953).

(3) C. Møller: *Kgl. Danske Vid. Selsk*, **23**, No. 1 (1945).

Such processes have been described in great detail (4) using a time-independent approa chand some attempts (5) have also been made to develop a time-dependent theory which has claimed attention (6), for example, in connection with the problem of exchange scattering. LOW (7) has prescribed the form $(\psi_f^{(-)}, \psi_i^{(+)})$ of the scattering matrix for rearrangement collisions. The present generalization leads exactly to the above form and is also in agreement with the work of LIPPMANN (6) and SUNAKAWA (8).

In dealing with collisions between parts of a many body system, we are asking for transition amplitudes from states describing the initially separate parts to states corresponding to finally separate parts which are, in general, different from the initial ones. Or using the terminology of scattering theory, we seek a transition matrix connecting states described in an interaction representation referred to the initial system to states in an interaction representation referred to the final system. By interaction representation we mean a description of the system with the time-dependence removed associated with a specified pair of free parts; and if the Hamiltonian of a system could be split as a sum, in more than one way, into parts corresponding to a pair of free systems and the interaction between them, it becomes possible to define different interaction pictures for the description of the system.

The temporal development of the state of a system is usually described independently of the initial state by regarding the evolution as the unfolding of a unitary transformation. It is of interest to observe the relationship between such unitary operators in different interaction pictures and to study the properties of that class of operators describing the time-development of the system from one picture to another. This is done in Section 2.

The S-matrix for rearrangement collisions is defined in Section 3 and the adiabatic hypothesis prescribed. The matrix elements appear in forms similar to those met with in scattering theory and provide agreement with LIPPMANN LOW and SUNAKAWA. In Section 4 we consider some examples.

2. – Transition operators for multichannel processes.

Consider a system whose Schrödinger equation is given by

$$i\hbar \frac{\partial \psi(t)}{\partial t} = H\psi(t) . \tag{2.1}$$

(4) A. M. LANE and R. G. THOMAS: *Rev. Mod. Phys.*, **30**, 257 (1958); where further references can be found.

(5) H. EKSTEIN: *Phys. Rev.*, **101**, 880 (1956).

(6) B. A. LIPPMANN: *Phys. Rev.*, **102**, 264 (1956).

(7) F. E. LOW: *Sumer Institute Lectures* (1959), Brandeis University.

(8) S. SUNAKAWA: *Prog. Theor. Phys.*, **24**, 963 (1960).

Let the Hamiltonian of the system be separable as

$$H = H_0^A + V^A \,, \tag{2.2}$$

where H_0^A describes the pair A of interacting parts of the system and V^A, the interaction between them.

The state $\Psi^A(t)$ of the system in interaction representation is defined by

$$\Psi^A(t) = \exp\,[iH_0^A t/\hbar]\,\psi(t) \tag{2.3}$$

and obeys the equation

$$i\hbar\,\frac{\partial \Psi^A(t)}{\partial t} = \exp\,[iH_0^A t/\hbar]\,V^A \exp\,[-\,iH_0^A t/\hbar]\,\Psi^A(t)\,. \tag{2.4}$$

The temporal development of the state can be represented by

$$\Psi^A(t) = U^A(t, t_0)\,\Psi^A(t_0)\,, \tag{2.5}$$

where $U^A(t, t_0)$ is the unitary transformation operator which obeys the differential equation

$$i\hbar\,\frac{\partial U^A(t, t_0)}{\partial t} = \exp\,[iH_0^A t/\hbar]\,V^A \exp\,[-\,iH_0^A t/\hbar]\,U^A(t, t_0)\,, \tag{2.6}$$

with the boundary condition $U^A(t_0, t_0) = 1$. The integral form of eq. (2.6) is given by

$$U^A(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} \exp\,[iH_0^A t'/\hbar]\,V^A \exp\,[-\,iH_0^A t'/\hbar]\,U^A(t', t_0)\,\mathrm{d}t'\,. \tag{2.7}$$

Suppose the same Hamiltonian H could also be separated as

$$H = H_0^B + V^B \tag{2.8}$$

corresponding to a different pair B of interacting parts and their interaction. Then we can define a corresponding interaction state $\Psi^B(t)$ as before and the time-development of the system could be expressed in terms of a unitary operator $U^B(t, t_0)$.

To make a comparison between the two unitary operators describing the evolution of the system, we observe that, starting with an initial state $\psi(t_0)$

at time t_0, the state $\psi(t)$ at t obtained via either representation must be the same

$$(2.9)\qquad \psi(t) = \exp[-iH_0^A t/\hbar]\Psi^A(t) = \exp[-iH_0^A t/\hbar]U^A(t,t_0)\exp[iH_0^A t_0/\hbar]\psi(t_0)\,.$$

Similarly

$$(2.10)\qquad \psi(t) = \exp[-iH_0^B t/\hbar]U^B(t,t_0)\exp[iH_0^B t_0/\hbar]\psi(t_0)\,.$$

Therefore

$$(2.11)\qquad \exp[-iH_0^A t/\hbar]U^A(t,t_0)\exp[iH_0^A t_0/\hbar] = U_S(t,t_0) = \\ = \exp[-iH_0^B t/\hbar]U^B(t,t_0)\exp[iH_0^B t_0/\hbar]\,,$$

and thus the transformation law for the operator $U(t, t_0)$ from one interaction picture to another is

$$(2.12)\qquad U^B(t,t_0) = \exp[iH_0^B t/\hbar]\exp[-iH_0^A t/\hbar]U^A(t,t_0)\cdot \\ \cdot\exp[iH_0^A t_0/\hbar]\exp[-iH_0^B t_0/\hbar]\,.$$

The operator $\lim\limits_{\substack{t\to+\infty\\ t_0\to-\infty}} U^A(t, t_0)$, the limits being taken using the adiabatic hypothesis, is the usual S-matrix describing scattering of the separated parts A and it connects a state φ_a^A describing the initially free parts A of the system with a state φ_b^A describing the same finally free parts. Similar is the role of $U^B(\infty, -\infty)$. The suffixes a or b denote the energy, relative momentum, angular momentum and any other attributes of the system. The problem of rearrangement collisions is clearly that of prescribing an S-matrix S^{BA} connecting a state φ_a^A of the initial free parts A, with a state φ_b^B of the finally separated parts B which are different from A. Therefore we seek an operator $U^{BA}(t, t_0)$ transforming a state in interaction representation A at time t_0 to a state at time t in interaction representation B.

Since

$$(2.13)\qquad \psi(t) = U_S(t,t_0)\psi(t_0)$$

or equivalently

$$(2.14)\qquad \Psi^B(t) = \exp[iH_0^B t_0/\hbar]U_S(t,t_0)\exp[-iH_0^A t_0/\hbar]\Psi^A(t_0)\,,$$

the relationship between the transformation operators $U^{BA}(t, t_0)$ and $U_S(t, t_0)$ is given by

$$(2.15)\qquad U^{BA}(t,t_0) = \exp[iH_0^B t/\hbar]U_S(t,t_0)\exp[-iH_0^A t_0/\hbar]\,.$$

Using (2.11), $U^{BA}(t, t_0)$ can be represented in terms of either $U^A(t, t_0)$ or $U^B(t, t_0)$ as follows:

$$U^{BA}(t, t_0) = \exp[iH_0^B t/\hbar]\exp[-iH_0^A t/\hbar]U^A(t, t_0) = \\ = U^B(t, t_0)\exp[iH_0^B t_0/\hbar]\exp[-iH_0^A t_0/\hbar]\,. \tag{2.16}$$

It is easily seen that $U^{BA}(t, t_0)$ satisfies also the property

$$U^{BA}(t, t_0) = U^{BC}(t, t_1)U^{CA}(t_1, t_0)\,, \tag{2.17}$$

where C denotes a possible interaction representation of the system. The differential equation satisfied by $U^{BA}(t, t_0)$ is

$$i\hbar\frac{\partial U^{BA}(t, t_0)}{\partial t} = \exp[iH_0^B t/\hbar]V^B\exp[-iH_0^B t/\hbar]U^{BA}(t, t_0)\,, \tag{2.18}$$

with

$$U^{BA}(t_0, t_0) = \exp[iH_0^B t_0/\hbar]\exp[-iH_0^A t_0/\hbar]\,.$$

It is easily seen that $U^{BA}(t, t_0)$ is unitary. It follows that $U^{BA}(t_0, t)$ operating on a « bra » state in interaction representation B at time t_0 takes it into the « bra » state at time t in interaction representation A and consequently obeys

$$-i\hbar\frac{\partial U^{BA}(t_0, t)}{\partial t} = U^{BA}(t_0, t)\exp[iH_0^A t/\hbar]V^A\exp[-iH_0^A t/\hbar]\,, \tag{2.19}$$

with

$$U^{BA}(t_0, t_0) = \exp[iH_0^B t_0/\hbar]\exp[-iH_0^A t_0/\hbar]\,.$$

The conjugate operator $U^{BA\dagger}(t, t_0)$ has a similar role.

3. – The S-matrix.

The S-matrix for rearrangement collisions can be defined as

$$S^{BA} = \lim_{\substack{t\to+\infty\\ t_0\to-\infty}} U^{BA}(t, t_0) \tag{3.1}$$

and it is necessary now to invoke the adiabatic hypothesis to obtain well-defined limits. This is accomplished by attaching a factor $\exp[-\varepsilon|t|]$ to

the interaction Hamiltonian, where ε is a small positive quantity which may be allowed to tend to the limit zero after a calculation is performed. An element $(\varphi_b^B, S^{BA}\varphi_a^A)$ of the S-matrix is now written as

$$(3.2)\qquad (\varphi_b^B, S^{BA}\varphi_a^A) = \lim_{\substack{t\to+\infty\\ t_0\to-\infty}} (\varphi_b^B, U^{BA}(t, t_0)\varphi_a^A) = (U^B(0, \infty)\varphi_b^B, U^A(0, -\infty)\varphi_a^A),$$

using (2.17) and (2.16). $U(0, \pm\infty)$ are the well-known Møller matrices, $\Omega^\pm$

$$(3.3)\qquad \Omega^\pm\varphi_a = \varphi_a + \frac{1}{E_a - H \pm i\varepsilon} V\varphi_a = \psi_a^{(\pm)}$$

Thus

$$(3.4)\qquad (\varphi_b^B, S^{BA}\varphi_a^A) = (\psi_b^{B(-)}, \psi_a^{A(+)}).$$

Eq. (3.4) is the definition of the generalized S-matrix in the stationary state formalism.

The integral equation form of (2.18) under the adiabatic hypothesis is

$$(3.5)\qquad U^{BA}(t, t_0) = \exp[iH_0^B t_0/\hbar]\exp[-iH_0^A t_0/\hbar] - \\ -\frac{i}{\hbar}\int_{t_0}^{t} \exp[iH_0^B t'/\hbar]V^B\exp[-\varepsilon|t'|]\exp[-iH_0^B t'/\hbar]U^{BA}(t', t_0)\,dt'.$$

The transition matrix element can now be written as

$$(3.6)\qquad (\varphi_b^B, S^{BA}\varphi_a^A) = \lim_{t_0\to-\infty} (\varphi_b^B, \exp[iH_0^B t_0/\hbar]\exp[-iH_0^A t_0/\hbar]\varphi_a^A) - \\ -\frac{i}{\hbar}\int_{-\infty}^{+\infty} (\varphi_b^B, \exp[iH_0^B t'/\hbar]V^B\exp[-\varepsilon|t'|]\exp[-iH_0^B t'/\hbar]U^{BA}(t', -\infty)\varphi_a^A)\,dt'.$$

The first term vanishes on energy integration in the limit $t_0\to-\infty$. However it should be noted that in the case of ordinary scattering ($H_0^B = H_0^A = H_0$), the first term reduces to (φ_b, φ_a) and is equal to δ_{ba}. Therefore

$$(3.7)\qquad (\varphi_b^B, S^{BA}\varphi_a^A)_{B\neq A} = (\varphi_b^B, T^{BA}\varphi_a^A) = \\ = -\frac{i}{\hbar}\int_{-\infty}^{+\infty} (\varphi_b^B, \exp[iH_0^B t'/\hbar]V^B\exp[-\varepsilon|t'|]\exp[-iH_0^A t'/\hbar]U^A(t', -\infty)\varphi_a^A)\,dt',$$

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using (2.16) together with a little manipulation. We now have

$$(3.8)\qquad T_{ba}^{BA} = -\frac{i}{\hbar}\,(\varphi_b^B,\, V^B \Psi_a^{A(+)}(E_b))\,,$$

where

$$(3.9)\qquad \Psi_a^{A(+)}(E) = \int_{-\infty}^{+\infty} \exp\,[i(E - H_0^A)t'/\hbar]\,\exp\,[-\,\varepsilon\,|t'|\,]\,U^A(t', -\infty)\varphi_a^A\,\mathrm{d}t'\,,$$

and thus satisfies the well-known Lippman-Schwinger equations

$$(3.10)\qquad \Psi_a^{A(\pm)}(E) = 2\pi\hbar\,\delta(E - E_a)\,\varphi_a^A + \frac{1}{E - H_0^A \pm i\varepsilon}\,V^A \Psi_a^{A(\pm)}(E)\,.$$

Removing the δ-function in the usual way as a common factor

$$(3.11)\qquad \Psi_a^{A(\pm)}(E) = 2\pi\hbar\,\delta(E - E_a)\,\psi_a^{A(\pm)}\,,$$

$\psi_a^{A(\pm)}$ is an eigenstate of the total Hamiltonian and is given by (3.3).

We have

$$(3.12)\qquad T_{ba}^{BA} = -\,2\pi i\,\delta(E_b - E_a)(\varphi_b^B,\, V^B \psi_a^{A(+)})\,.$$

In a similar way one can show that the transition matrix element T_{ba}^{BA} is also given by

$$(3.13)\qquad T_{ba}^{BA} = -\,2\pi i\,\delta(E_b - E_a)(\psi_b^{B(-)},\, V^A \varphi_a^A)$$

and thus the transition matrix element $\boldsymbol{T}_{ba}^{BA}$ on the energy shell is

$$(3.14)\qquad \boldsymbol{T}_{ba}^{BA} = (\varphi_b^{B(-)},\, V^B \psi_a^{A(+)}) = (\psi_b^{B(-)},\, V^A \varphi_a^A)\,.$$

It is worth pointing out a striking feature of (3.14) which is not present if $V^A = V^B = V$. $\psi^{(+)}$ and $\psi^{(-)}$ solutions imply that the perturbation might have acted any number of times and represent a linear combination of states off and on the energy shell. If $V^A = V^B = V$ we can convince ourselves that the projection of $\psi_a^{(+)}$ on φ_b, or φ_a on $\psi_b^{(-)}$ through *one* perturbation is equal to the matrix element $(\psi_b^{(-)}, \psi_a^{(+)})$ on the energy shell. (3.14) *implies that* V^B *has acted once and* V^A *any number of times or* V^A *once and* V^B *any number of times. That these two should be equal to* (3.4) *even when* $V^A \neq V^B$ *is significant since* (3.4) *implies that both* V^A *and* V^B *could have acted any number of times.*

The above relations (3.12) and (3.13) can also be obtained from (3.4) using the technique of operator algebra [7].

4. – Examples.

The above considerations provide a clear background for the application of the final state boundary condition in connection with problems like deuteron stripping [9]. Adopting an obvious notation the Hamiltonian, H_0^A of the free initial parts is

$$H_0^A = H(Z, N) + H(d) = H(Z, N) + T_n + T_p + V_{np}\,, \tag{4.1}$$

where Z and N denote respectively the number of protons and neutrons in the struck nucleus and T_n, T_p are the kinetic energy operators. The interaction V^A between the initial parts is

$$V^A = V_{d,(Z,N)} = V_{n,(Z,N)} + V_{p,(Z,N)}\,. \tag{4.2}$$

In the final state B we have a nucleus $(Z, N+1)$ and a proton

$$H_0^B = H(Z, N+1) + H(p) = H(Z, N) + T_n + V_{n,(Z,N)} + T_p\,, \tag{4.3}$$

and

$$V^B = V_{p,(Z,N+1)} = V_{p,(Z,N)} + V_{np}\,. \tag{4.4}$$

Evidently

$$H_0^A + V^A = H = H_0^B + V^B$$

and the matrix element for the process can be written using (3.14) as

$$T_{ba}^{BA} = (\chi_b^B, [V_{p,(Z,N)} + V_{np}]\psi_a^{A(+)})\,,$$

where

$$\psi_a^{A(+)} = \chi_a^A + \frac{1}{E_a - H + i\varepsilon} V^A \chi_a^A\,,$$

and χ_a^A and χ_b^B are eigenstates of the free Hamiltonians (4.1) and (4.3) respectively. Introducing the final state boundary condition following WATSON [10], the matrix element can be written as

$$T_{ba}^{BA} = (\chi_b^{(-)}, V_{np}\psi_a^{A(+)})\,, \tag{4.5}$$

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[9] N. C. FRANCIS and K. M. WATSON: *Phys. Rev.*, **93**, 313 (1954).

[10] K. M. WATSON: *Phys. Rev.*, **88**, 1163 (1952).

where

$$\chi_b^{(-)} = \chi_b^B + \frac{1}{E_b - H_0^B - i\varepsilon} V_{\mathrm{p},(Z,N)} \chi_b^{(-)}, \tag{4.6}$$

is a scattering solution of $H_0^B + V_{\mathrm{p},(Z,N)}$.

Replacing $\psi_a^{A(+)}$ in (4.5) by ψ_e,

$$\psi_e = \chi_a^A + \frac{1}{E_a - H_0^A + i\varepsilon} \mathscr{V}_{ON} \psi_e, \tag{4.7}$$

the matrix element is interpreted (9) as representing the distortion of the incident deuteron wave function by the nuclear optical potential, $\mathscr{V}_{ON}$ and a subsequent impulsive break-up of the deuteron by the neutron-proton force; if the distortion is small, the expression leads to Born approximation.

Similar considerations apply to the case of deuteron break up (11). The problem is different from the previous one only in that the final state now consists of a free proton and a free neutron and the nucleus (Z, N)

$$H_0^B = H(Z, N) + T_\mathrm{n} + T_\mathrm{p} \tag{4.8}$$

and

$$V^B = V_{\mathrm{p},(Z,N)} + V_{\mathrm{n},(Z,N)} + V_{\mathrm{np}}. \tag{4.9}$$

As before the matrix element for the process is

$$(\chi_b, \boldsymbol{T}^{BA}\chi_a) = (\chi_b^{(-)}, V_{\mathrm{np}}\psi_a^{A(+)}), \tag{4.10}$$

where

$$\chi_b^{(-)} = \chi_b + \frac{1}{E_b - H_0^B - i\varepsilon} (V_{\mathrm{p},(Z,N)} + V_{\mathrm{n},(Z,N)}) \chi_b^{(-)}. \tag{4.11}$$

Replacing $V_{\mathrm{p},(Z,N)}$ and $V_{\mathrm{n},(Z,N)}$ by the optical potentials $\mathscr{V}_\mathrm{p}$ and $\mathscr{V}_\mathrm{n}$, one gives a physical interpretation to the matrix element that after the deuteron break-up, the neutron and proton separately undergo elastic scattering in the nuclear optical potential. A similar interpretation could also be given for the final state in (4.5) but it is peculiar that the proton moves in an optical potential due to the nucleus (Z, N) and not $(Z, N+1)$ which is formed. However in the successful Butler theory, the proton is assumed not to enter the nucleus due to the Coulomb barrier in which case the problem does not arise.

(11) J. E. YOUNG: *Phys. Rev.*, **116**, 1201 (1959).

* * *

It is a pleasure to thank Mr. K. VENKATESAN and Mr. N. R. RANGANATHAN for interesting discussions and one of us (G.R.) is grateful to the Council of Scientific and Industrial Research (India) for the award of a Junior Research Fellowship.

RIASSUNTO (*)

Presentiamo una teoria, con dipendenza temporale, delle collisioni di riassestamento, che è una semplice generalizzazione della ben nota teoria dello scattering. Otteniamo la matrice S per questi processi e troviamo che gli elementi di matrice appaiono in forme simili a quelle della teoria dello scattering. Come illustrazione discutiamo alcune applicazioni elementari.

(*) *Traduzione a cura della Redazione.*

2.22

IL NUOVO CIMENTO VOL. XXI, N. 1 1° Luglio 1961

The Resonant Pion-Pion Model for the Nucleon Structure.

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(ricevuto il 10 Maggio 1961)

Summary. — A theoretical model based on strong π-π interaction (1) is compared with experimental data on nucleon form factors and the determination of the parameters is discussed. The qualitative evaluation of the positions of the two and three pion resonances given in (1) is not in disagreement with the existing data, but a wide range of values are allowed for the parameters. The situation is not clear in view of some difficulties in the interpretation of the neutron form factors.

1. – Introduction.

Recently a model for the electromagnetic structure of the nucleon was proposed [1], based on the hypothesis of a strong correlation between the pions.

Let us summarize the qualitative discussion contained in A which justifies the assumptions of the model.

1) The rapid variation of the nucleon form factors is due to a strong correlation between the pions.

2) At low momentum transfers ($\leqslant 10$) F_{1p} and F_{2p} were interpolated by CLEMENTEL and VILLI [2] with the following expression:

$$F_{1p}(t) \simeq F_{2p}(t) \equiv F(t) = -0.2 + \frac{1.2}{1 - t/22}, \tag{1}$$

[1] S. BERGIA, A. STANGHELLINI, S. FUBINI and C. VILLI: *Phys. Rev. Lett*, **6**, 367 (1961), quoted in the following as A; notations are the same used in this paper; the energies are measured in m_π (pion mass).

[2] E. CLEMENTEL and C. VILLI: *Nuovo Cimento*, **4**, 1207 (1958).

BOWCOCK, COTTINGHAM and LURIÉ [3] have interpreted this result as if the isovector part of the nucleon structure were dominated by a two-pion $T=1$, $J=1$ resonance at an energy $E_R \simeq \sqrt{22} \simeq 4.7$.

3) The isoscalar part must be important in the same momentum transfer region, in order to give a vanishing neutron charge radius.

4) Experiments at higher momentum transfers [4] show that the proton charge form factor deviates substantially from Clementel and Villi's formula (eq. (1)). This deviation is less striking for the magnetic form factor.

In A these deviations have been assigned to the isoscalar part of the nucleon structure, which enters mainly in the charge form factor due to the small isoscalar magnetic moment $g_S = (g_p + g_n)/2 = -0.06$.

It was also argumented that the isoscalar part is probably formed of a large constant part and a part which vanishes much faster than the isovector part.

The last contribution was interpreted as a resonance in the $T=0$, $J=1$ three-pion state with an energy lower than that assigned to the $T=1$, $J=1$ resonance.

In order to give a qualitative support to the last idea let us suppose that two different poles contribute one to the charge structure and the other to the magnetic structure of the nucleon. Evidently these poles are only to be thought of as « equivalent poles » in the sense that they summarize the behaviour of different combinations of the isoscalar and the isovector contributions.

Let us construct the following quantities:

$$A_1(t) = \frac{F_{1p}(t) - 1}{t},$$

$$A_2(t) = \frac{F_{2p}(t) - 1}{t}.$$

If one pole only is present, we have:

$$R_1(t) = \frac{1}{A_1(t)} = \frac{t - t_1}{a_1},$$

$$R_2(t) = \frac{1}{A_2(t)} = \frac{t - t_2}{a_2}.$$

(3) J. BOWCOCK, W. N. COTTINGHAM and D. LURIÉ: *Phys. Rev. Lett.*, **5**, 386 (1960).

(4) R. HOFSTADTER, F. BUMILLER and M. CROISSIAUX: *Phys. Rev. Lett.*, **5**, 263 (1960). Experimental data are taken from: *a*) R. HOFSTADTER: *Ann. Rev. Nucl. Sci.*, **7**, 231 (1957) and from ref. (4) for higher momentum transfers. For the elder set of data arbitrarily a small systematic error has been added to the experimental errors given in (*a*) for values of t between 7 and 12, in order to take into account the possibility of an effective separation of F_{1p} and F_{2p} also in this region; for other data errors of the order of magnitude given in ref. (4) have been taken.

In Fig. 1 we have plotted the experimental R_1 and R_2 together with Clementel and Villi's formula.

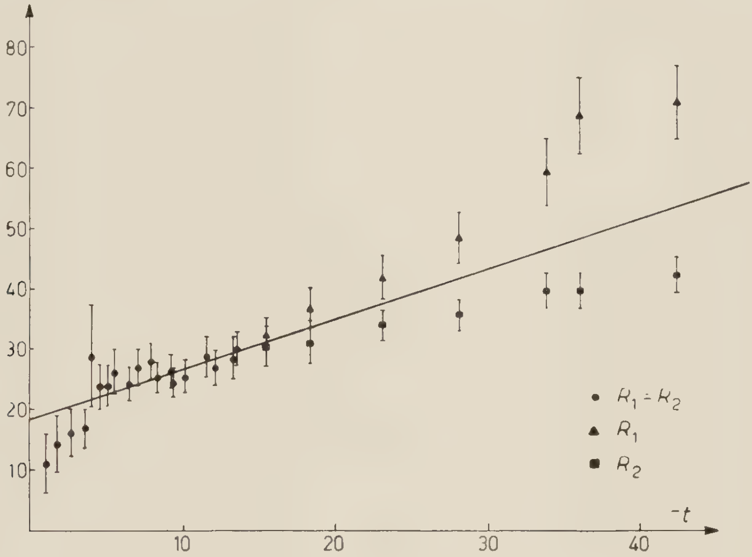

Fig. 1. – $R_1 = t/(F_{1p}(t) - 1)$ and $R_2 = t/(F_{2p}(t) - 1)$ are plotted against t together with Clementel and Villi's formula (I).

At low momentum transfers R_1 and R_2 have not been experimentally separated but from what one knows are certainly of the same order of magnitude and so:

$$\frac{t_1}{a_1} \simeq \frac{t_2}{a_2}\,.$$

On the other hand at higher momentum transfers $R_1 R_2$, which leads to:

$$\frac{1}{a_1} > \frac{1}{a_2}\,,$$

from which $t_1 < t_2$.

This means that the pole which could explain the charge form factor is different and lower than the one which explains the magnetic form factor. Due to the small contribution of the isoscalar part to the magnetic form factor, we think that t_2 is a good representative of a pole t_V in the isovector part and t_1 comes from a combined effect of $t_2 \simeq t_V$ and a pole at t_S of the isoscalar part, which has to be smaller than t_V in order to decrease the value of t_1 appreciably under t_2.

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In this manner we have given a stronger support to the hypothesis contained in A.

From the above considerations the following expressions for the nucleon form factors were proposed [*]:

$$
(2)\qquad \left\{
\begin{aligned}
F_{1V} &= (1 - c_V) + \frac{a_V}{1 - t/t_V}\,; \qquad & F_{1S} &= (1 - a_S) + \frac{a_S}{1 - t/t_S}\,,\\
F_{2V} &= (1 - b_V) + \frac{b_V}{1 - t/t_V}\,; \qquad & F_{2S} &= (1 - b_S) + \frac{b_S}{1 - t/t_S}\,,
\end{aligned}
\right.
$$

where t_V is expected to be of the order of 22 and t_S much smaller, let us say 10.

The scope of this paper is to discuss the determination of the parameters appearing in eq. (2), using the experimental data.

We wish to show that the qualitative evaluation of the resonance positions is not in disagreement with the existing data.

A very important point is to check if the isoscalar part is dominated by a resonance ($t_S > 9$) or by a bound state ($t_S < 9$) and if there are any chances to identify it with the « particle » observed by CROWE and co-workers [5], as it was suggested in A.

We shall show that the existing data are not sufficient for an unambiguous determination of the parameters, *i.e.* that a wide range of values are allowed.

2. – Discussion of the experiments.

We wish to discuss the different sources of experimental information about the nucleon form factors.

The preceding qualitative discussion of the model is mainly based on the experimental data on proton form factors.

The reason is that these form factors are the best known and their derivation from the experiments is straightforward.

We have tried to use the proton form factors in order to determine the parameters of the model, but, as we shall show this is not possible because of the uncertainties of the experimental data.

The best information on the neutron is the knowledge of the charge mean square radius, obtained from neutron scattering on atomic electrons, which turns out to be very small:

$$\frac{r_{1n}^2}{6} \qquad 5\cdot 10^{-4}\,.$$

(*) Formulas (2) were calculated in the case of infinitely narrow width. For a finite, but still narrow resonance one obtains corrections of the order of $[\Gamma/t_R]^2$ which in the actual case of the $T=J=1$ resonance are very small (a few per cent).

(5) A. ABASHIAN, N. E. BOOTH and K. M. CROWE: *Phys. Rev. Lett.*, **5**, 258 (1960).

For our model this means that:

$$\frac{1}{2}\left(\frac{a_S}{t_S}-\frac{a_V}{t_V}\right) < 5\cdot 10^{-4}\,.$$

On the other hand the proton charge mean square radius is:

$$\frac{r^2_{1p}}{6} \simeq 5\cdot 10^{-2}\,,$$

with an uncertainty of the order of 10%.

Thus one has:

$$\frac{1}{2}\left(\frac{a_S}{t_S}+\frac{a_V}{t_V}\right) \simeq 5\cdot 10^{-2}\,.$$

This means that the difference between the two quantities a_S/t_S and a_V/t_V is certainly smaller that their errors and so we can safely put:

$$\frac{1}{2}\frac{a_S}{t_S}=\frac{1}{2}\frac{a_V}{t_V}=a\,, \tag{3}$$

and eliminate in this was one parameter from the model.

The other sources of information about the structure are:

a) elastic electron-deuteron scattering;

b) electrodisintegration of the deuteron;

c) electroproduction of pions from protons.

The derivation of the neutron form factors from the above experiments is not so simple as for the proton case. In effect one needs the theory of the strong interacting particles involved, like the deuteron, the pion-nucleon and the nucleon-nucleon systems, and the upper bounds of the errors coming from the approximation introduced are not very well known. From *a*) one obtains directly the ratio (6)

$$\frac{F_{2S}}{F_{1S}}=\left(\frac{0.06\pm 0.09}{-0.12}\right),$$

in the range of momentum transfers $5\leqslant t\leqslant 10$.

The «theoretical error» is estimated of the order of 0.13.

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(6) J. I. FRIEDMANN, H. W. KENDALL and P. A. M. GRAM: *Phys. Rev.*, **120**, 992 (1960).

This result can also be presented as

$$\frac{F_{2p}}{F_{2n}} = (0.91 \pm 0.05)\,.$$

For reaction *b*), there are many experiments in a wide range of momentum transfers ([7-9]) as for the proton. In order to deduce the neutron cross-sections one uses the impulse approximation neglecting the final state interaction and the interference between proton and neutron contributions ([10]).

Rough evaluations of these effects give corrections which might decrease the deuteron cross-sections by $(10 \div 15)\,\%$. The neutron cross-section turns out to be only a minor part of the deuteron one (less than $\frac{1}{3}$) and so any error is enhanced for it.

The results obtained by HOFSTADTER and co-workers ([8,9]) using this method can be summarized as follows:

1) F_{2n} is larger than F_{2p},

2) F_{1n} is small and positive

and the root mean square radii of the charge and magnetic moment distributions of proton and neutron are:

$$a_{1p} = 0.85\,\text{fermi}\,; \qquad a_{1n} = 0\,; \qquad a_{2p} = 0.94\,\text{fermi}\,; \qquad a_{2n} = 0.76\,\text{fermi}\,.$$

The neutron cross-sections are also given by WILSON and co-workers ([7]). We have calculated the WILSON cross-sections using the Hofstadter form factor and we have found that the calculated cross-sections are larger than the experimental ones by a large factor ([*]).

From *c*) one obtains the magnetic root mean square radius of the neutron ([11]) which is of the order of 1.0 fermi, but it is not easy to evaluate the approximations of the theory given by FUBINI, NAMBU and WATAGHIN ([12]).

We see that between different neutron experiments there are inconsistencies, which are mainly due to the poorness of the theory involved.

(7) D. N. OLSON, H. F. SCHOPPER and R. R. WILSON: *Phys. Rev. Lett.*, **6**, 286 (1961).

(8) R. HOFSTADTER, C. DE VRIES and R. HERMAN: *Phys. Rev. Lett.*, **6**, 290 (1961).

(9) R. HOFSTADTER and R. HERMAN: *Phys. Rev. Lett.*, **6**, 293 (1961).

(10) V. Z. JANKUS: *Phys. Rev.*, **102**, 1586 (1958); R. BLANKENBECLER: *Phys. Rev.*, **111**, 1684 (1958); A. GOLDBERG: *Phys. Rev.*, **112**, 618 (1958).

(*) It seems that the deduction of the neutron cross sections from the deuteron ones is not correct. We thank Prof. R. HOFSTADTER for this information.

(11) G. G. OHLSEN: *Phys. Rev.*, **120**, 584 (1960).

(12) S. FUBINI, Y. NAMBU and V. WATAGHIN: *Phys. Rev.*, **111**, 329 (1958).

In order to overcome this difficulty, we have thought to base our analysis on the proton form factors first and then use of the neutron only the common features of all the experiments discussed, which are the low momentum transfer results.

In particular we have choosen as representative the elastic scattering results (reaction a)).

3. – Determination of the parameters.

Let us write the expressions for the form factors given by the model once condition (3) is taken in account:

$$\text{(4)} \quad \begin{cases} F_{1p} = 1 + a\left(\dfrac{t_S}{t_S - t} + \dfrac{t_V}{t_V - t}\right)t\,; & F_{1n} = -\,a\left(\dfrac{t_V}{t_V - t} - \dfrac{t_S}{t_S - t}\right)t\,, \\ F_{2p} = 1 + \left(\dfrac{P_S\, b_S}{t_S - t} + \dfrac{P_V\, b_V}{t_V - t}\right)t\,; & F_{2n} = 1 + \left(\dfrac{\mathcal{N}_S b_S}{t_S - t} - \dfrac{\mathcal{N}_V b_V}{t_V - t}\right)t\,, \end{cases}$$

$$P_S = \frac{g_S}{g_p} = -\,0.0335\,; \qquad n_S = \frac{g_S}{g_n} = \quad 0.031\,,$$

$$P_V = \frac{g_V}{g_n} = 1.0335\,; \qquad n_V = \frac{g_V}{g_n} = -\,0.969\,.$$

Eq. (4) contain 5 independent parameters, which shall be determined by requesting the best fit of the experimental data using least squares method.

Let us first consider the proton problem.

We have made full use of the fact that only t_S and t_V enter both in the charge and in the magnetic form factor, while a is present in the charge form factor and b_S, b_V in the magnetic form factor only.

Let us suppose to know t_S and t_V, *i.e.* the position of the two resonances. The problem of determining a, b_S, b_V is similar to that of the Chew and Low extrapolation (with a known extrapolating function), for the determination of the residue of some poles.

We have constructed the quantities:

$$\text{(5)} \qquad A_1 = \frac{F_{1p} - 1}{t} \qquad \text{and} \qquad A_2 = \frac{F_{2p} - 1}{t} \qquad \text{(see Fig. 2)}\,.$$

The quantities $A_1(t_S - t)(t_V - t)$ and $A_2(t_S - t)(t_V - t)$ are linear functions of t and the determination of the parameters from the experiments are obvious.

Evidently we don't know the t_S and t_V and so we are obliged to ask the

experiments to make a selection among the possible values, *i.e.* which pair of t_S and t_V gives the best fit.

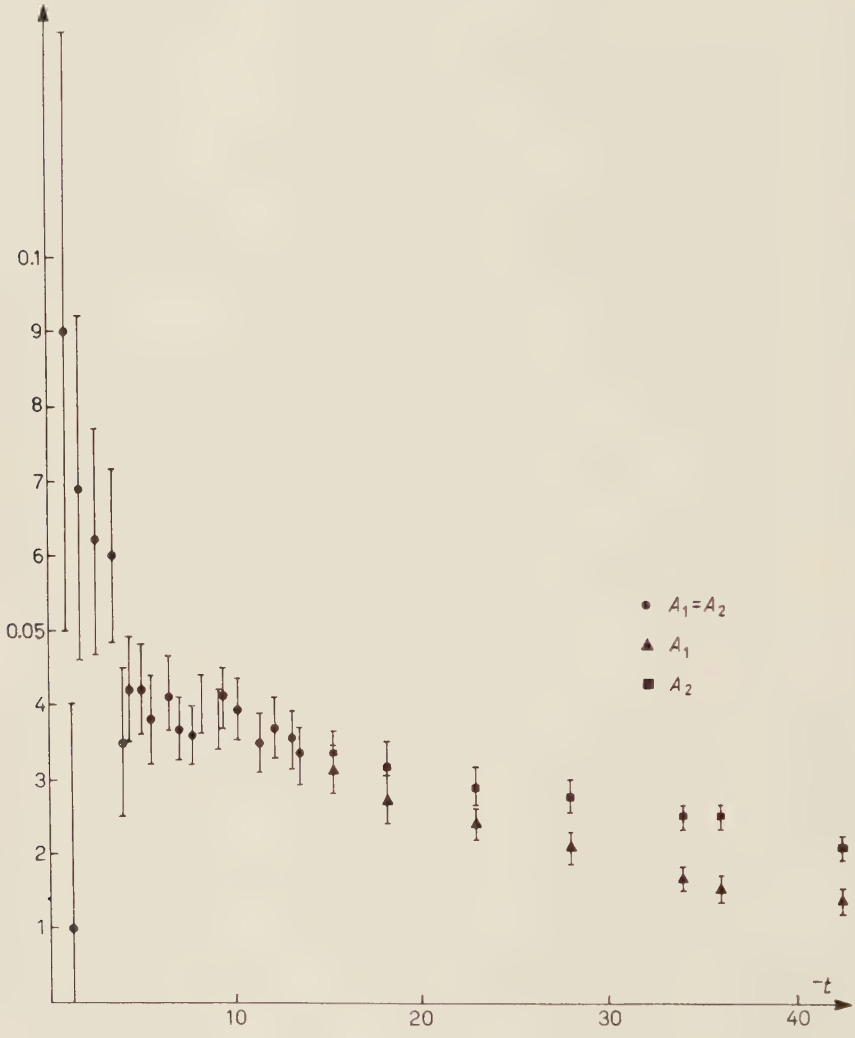


Fig. 2. – Experimental $A_1 = (F_{1p}(t) - 1)/t$ and $A_2 = (F_{2p}(t) - 1)/t$ are plotted against t.

The method we have followed has been to fix a pair of t_S and t_V and determine the corresponding values of a, b_S and b_V, which, in this sense, are functions of t_S and t_V.

The values of these parameters are sufficiently well determined by the experiments on proton form factors. Values of them as functions of t_S and t_V are plotted in Fig. 3 through 5.

For each pair of t_s and t_v one calculates:

$$M_1 = \\ = \sum_i \left(\frac{F^i_{1p} - F_{1p}(a, t_S, t_V, t_i)}{\Delta F^i_{1p}} \right)^2,$$

and the analogous quantity for F_{2p}, say M_2.

By considering the sum $M_1 + M_2$ we have seen that the proton itself does not make a clear-cut selection for definite values of t_s and t_v.

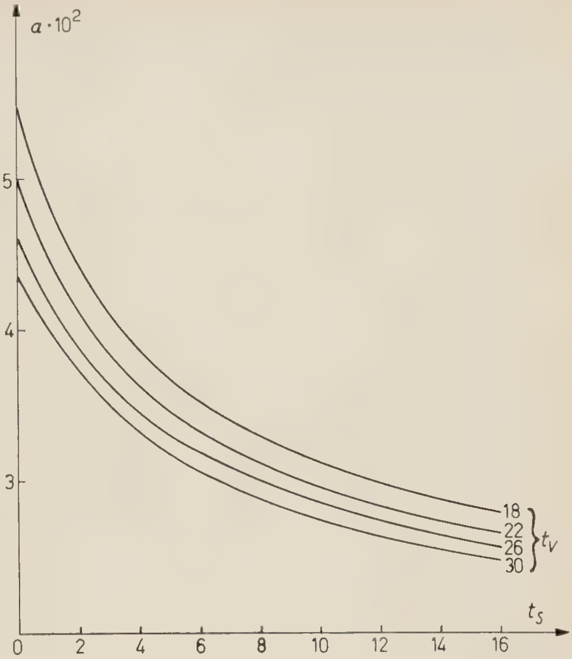
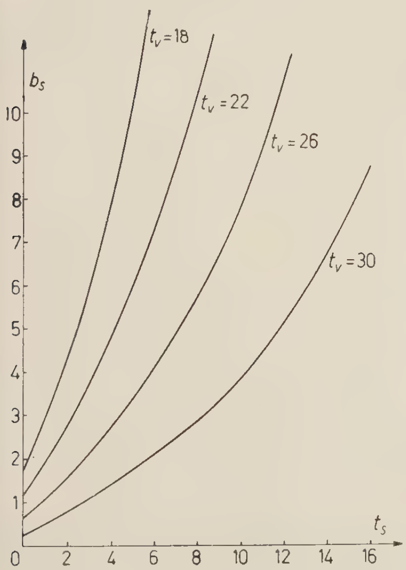


Fig. 3. – a as a « function » of t_s is plotted for some values of t_v.

We think that one of the reasons why it is not possible to determine the parameters of the model from the proton experiments is that the $t = 0$ values of A_1 and A_2 are not well determined by the experiments as one sees from Fig. 3; these values are proportional to the mean square radii of the charge and magnetic moment distributions of the proton and are quite dependent on the parameters of the model, *i.e.* on t_s and t_v. Accurate experiments for t less than 5 are highly desirable. In order to restrict the range of possible values for t_s and t_v, we have used the four experimental points for $F_{2S}/F_{1S} = \alpha$ as given in ref. [6].



Using the values of the parameters obtained from the proton fit, we have calculated:

$$M_3 = \sum_i \left(\frac{\alpha_i^{\exp} - \alpha(a, b_S, t_S, t_i)}{\Delta\alpha_i^{\exp}} \right)^2.$$

Fig. 4. – b_S as a « function » of t_S is plotted for some values of t_V.

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A «map» giving the value of the sum $M_1+M_2+M_3$ as function of t_s and t_v is given in Fig. 6.

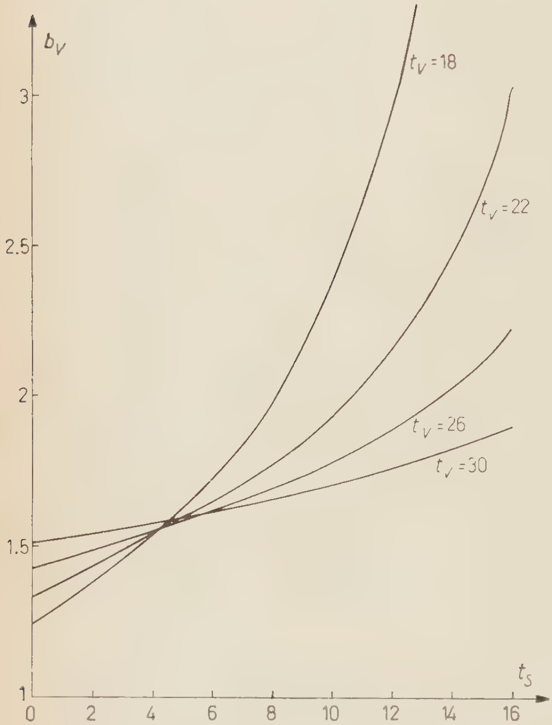

Fig. 5. – b_v as a «function» of t_s is plotted for some values of t_v.

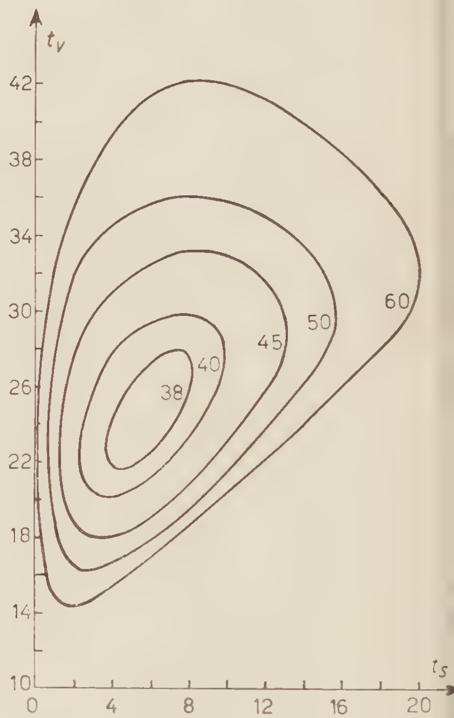

Fig. 6. – The sum $M_1+M_2+M_3$ is given for pairs of t_s and t_v in a region around the minimum; some level curves are drawn.

As one sees the best values are:

$$t_s \simeq 6 \; ; \qquad t_v \simeq 25 \; .$$

With this pair of t_s and t_v we have:

$$a = 3.23 \cdot 10^{-2}$$

$$b_s = 4.75$$

$$b_v = 1.63$$

and thus

$$a_s = 0.39$$

$$a_v = 1.62$$

(notice that $a_v \simeq b_v$).

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However statistically a wide range of values around the central ones are allowed.

Some qualitative features of these solutions are remarkable:

1) t_s is smaller than t_v;

2) in this range we have a_v, b_v, positive, b_s positive apart for very large values of t_v.

4. – Discussion of the results.

We can conclude that the model is not contradictory with the experiments we have choosen as representative, but the parameters are not very well determined. In Fig. 7 we show the fits for the charge and magnetic moment proton form factors, which are fairly satisfactory.

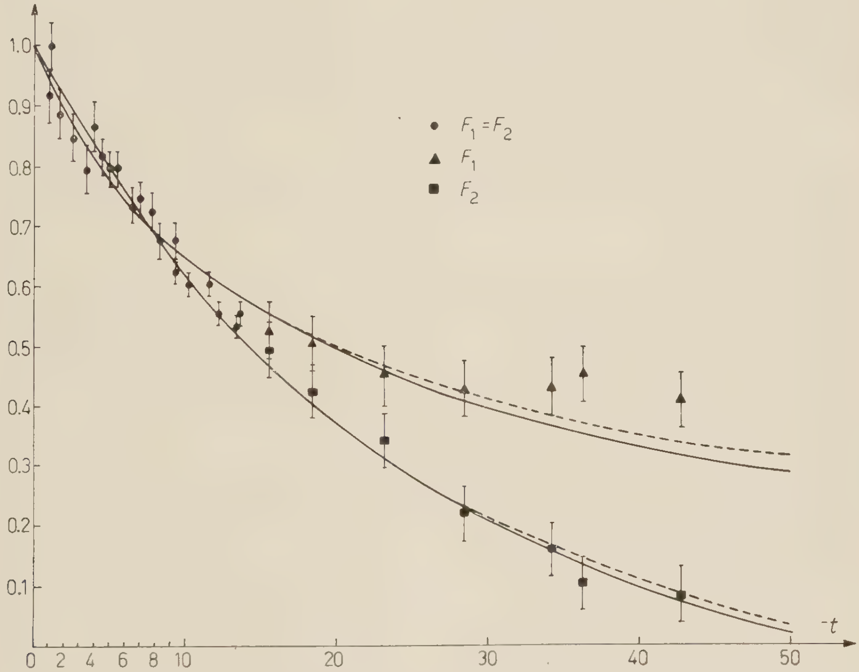

Fig. 7. – Experimental proton charge and magnetic form factors. The full line is a representative of the proton charge and magnetic form factors calculated with our best set of parameters and the broken line with the one corresponding to $t_v = 22$.

The situation for the form factors does not allow to discriminate between $t_s \lessgtr 9$ even if $t_s < 9$ is preferred.

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As to the neutron form factors, the qualitative features for a wide range of values for t_S and t_V around the best values are that F_{1n} is positive and F_{2n} is always smaller than F_{2p} (Fig. 8).

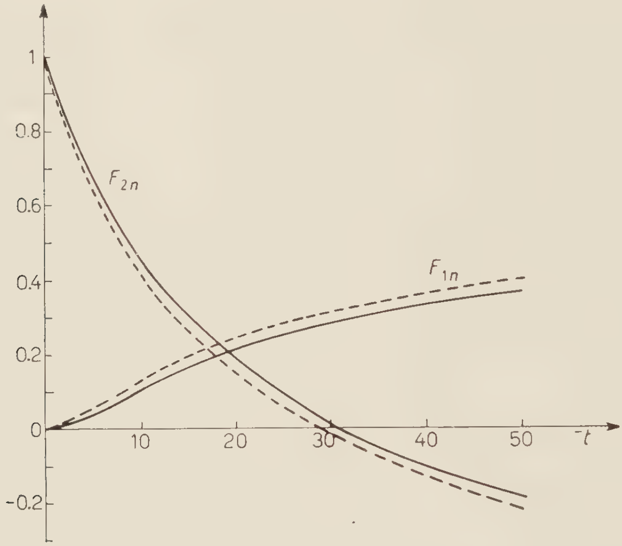

Fig. 8. – Neutron charge and magnetic form factors as given by our best set of parameters (full line) and with the one corresponding to $t_V = 22$ (broken line).

If we calculate the neutron cross-sections by our form factors, we cannot reproduce the experimental results of HOFSTADTER and co-workers [8].

As we have already discussed the deduction of neutron cross-sections from deuteron cross-sections is very delicate and it may be that our results are not in contradiction with the experiments once any sources of experimental and theoretical errors are taken into account.

A similar analysis on the basis of eq. (2) has been carried out by HERMAN and HOFSTADTER [9], who have determined the parameters by trying a best fit of both their proton and neutron data.

In order to fit the neutron data, b_S is forced to assume negative values and the set of parameters of HERMAN and HOFSTADTER is the following:

$$t_S \simeq 9.2\,, \qquad t_V \simeq 20\,,$$

$$b_S \simeq -3\,, \qquad b_V \simeq 1.2\,,$$

$$a_S \simeq 0.56\,, \qquad a_V \simeq 1.2\,.$$

As one sees the values of the parameters are not in disagreement with ours except for b_S, *i.e.* our mean square radius for the neutron magnetic distri-

bution is larger than that of HERMAN and HOFSTADTER. We wish to stress however that with this set of parameters a less satisfactory fit of proton form factors is obtained.

We think however that the situation is rather delicate and the problem is still open.

It is quite important to remark that the model proposed in A is phenomenological in its nature. Its value does not lie mainly in giving a possible good fit of the experimental nucleon form factors, but in the fact that the constants introduced are related with others appearing in different processes. This means that the two- and three-pion resonances, in the same positions and with the same quantum numbers, have to be discovered in processes like pion production in pion nucleon collisions, etc. Now there is some evidence for a $T=1$, $J=1$ resonance from experiments of pion production at $t_V \simeq 22$ which value is not far away from our best one [13] [*].

If we fix t_V to be equal to 22, the most remarkable fact is that t_S turns out to be less than 9 and values greater than 9 are very unlikely; the best values are around 5.

The only object of this kind which seems to have been observed is the Crowe particle and it seems reasonable to us to make this identification.

By choosing both $t_V=22$ and $t_S=5$, the remaining parameters are fixed as follows:

$$a = 3.46\cdot 10^{-2}$$

$$b_S = 5.90$$

$$b_V = 1.60$$

and thus:

$$a_S = 0.35$$

$$a_V = 1.52\,.$$

In Figs. 7 and 8 we have plotted also the curves we obtain with this choice, which turn out to be not very far from our best fitting ones.

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(13) E. PICKUP, F. AYER and E. O. SALANT: *Phys. Rev. Lett.*, **4**, 474 (1960); J. G. RUSHBROOK and D. RADOJICIC: *Phys. Rev. Lett.*, **4**, 567 (1960); J. A. ANDERSON VO. X. BANG, P. G. BURKE, D. D. CARMONY and N. SCHMITZ: *Phys. Rev. Lett.*, **6**: 365 (1961).

(*) *Note added in proof.* - Recent experiments on pion production (D. STONEHILL, C. BALTAY, H. COURANT, W. FICKINGER, E. C. FOWLER, H. KRAYBILL, J. SANDWEISS J. SANFORD and H. TAFT: *Phys. Rev. Lett.*, **6**, 624 (1961); A. R. ERWIN, R. MARCH, W. D. WALKER and E. WEST: *Phys. Rev. Lett.*, **6**, 628 (1961)) confirm the existence of a $T=J=1$ pion-pion resonance, whose position is $t_V \simeq 27$ and the full width of the order of $5m\pi^2$.

So we can conclude that it is possible to choose parameters compatible both with the set of nucleon form factors data we have select and with the experimental information (if reliable) on two- and three-pion resonant or bound states. However in view of the difficulties outlined, both theoretical and experimental information is needed in order to clarify the situation. An accurate and complete set of experimental data together with some improvement of the underlying theory are likely to give the definite answer to the several aspects of the problem.

* * *

We are deeply indebted to Prof. S. Fubini for many helpful discussions and advice. We wish to thank Prof. R. Hofstadter for a useful corrispondence we have had with him.

The assistance of Mr. I. Maganzani for the programming work at the I.B.M. 650 of the «Centro Calcoli» of the University of Bologna is gratefully recognized.

RIASSUNTO

Si usano i dati sperimentali per determinare i parametri di un modello per i fattori di forma dei nucleoni basato su una forte interazione pione-pione. Si mostra che le posizioni delle risonanze negli stati di due e tre pioni stimate in A sono in accordo qualitativo coi dati esistenti. Tuttavia un'ampia serie di valori per le posizioni delle risonanze è compatibile con gli esperimenti. La situazione non è chiara a causa di difficoltà nell'interpretazione teorica e sperimentale dei fattori di forma del neutrone.

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Neutral Scalar σ-Meson and the Mass Difference between Muon and Electron (*).

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Summary. — The self-mass arising from the interaction of a leptonic field with a scalar neutral σ-meson is calculated in perturbation theory to examine the possibility that the electron may be regarded as a light muon because the sign of the self-mass is negative. Further, the consequences of a model in which a leptonic field interacting with the σ-meson can describe the electron and muon as eigenstates are studied.

1. – Introduction.

One of the most puzzling features of the leptons (the electron and muon) is the high symmetry of all their known interactions in the face of their large mass difference. If one wants to attribute the mass differences between elementary particles to their interactions, then the large mass difference between the electron and muon suggests the existence of either a new interaction or a new particle, hitherto undiscovered, whose purpose is to provide the explanation of the mass difference. Of course, the whole approach may be wrong, in which case some drastic change would be required.

It is indeed distasteful to introduce a new particle only for the purpose of explaining one mass difference but there seems to be a *raison d'être* for a neutral scalar σ-meson if its existence would not change anything that is

(*) Work done under the auspices of the U. S. Atomic Energy Commission and the U. S. Air Force through contract no. AF 61(052)434.

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already known about elementary particles. The σ-meson was postulated by SCHWINGER on the basis of general arguments on the connection between isotopic spin, charge, and hypercharge ([1]). It is scalar under all operations in the three-dimensional isotopic space and in space-time. It further has the interesting feature that, to the second order in perturbation theory, its scalar interaction with a leptonic field gives rise to a self-mass, whose sign is negative. Then if the leptonic field $\psi(x)$ with mass m can describe the muon and the electron states, one obtains the condition that the mass M_e of the electron and the mass M_μ of the muon are less than the mass m of the leptonic field $\psi(x)$. Hence, when the limiting procedure of GELL-MANN and LOW ([2]) is applied to the integral equation of the propagator $S'_F(x-y)$ of $\psi(x)$, where

$$S'_F(x-y) = S_F(x-y) + \int \mathrm{d}z \int \mathrm{d}\omega\, S_F(x-\omega) G(\omega - z) S'_F(z-y)\,, \tag{1}$$

the inhomogeneous term (the free propagator) $S_F(x-y)$ drops out because it does not contain any frequencies as low as that corresponding to the masses M_e and M_μ.

One then obtains an integral equation for the leptonic wave function ([3]) $\chi_\alpha(x) = \langle 0 | \psi(x) | \alpha \rangle$, where the one-particle state $|\alpha\rangle$ is an eigenstate of the momentum four-vector P_μ with eigenvalues p whose mass M $(p^2 = -M^2)$ will be that of the M_e and M_μ. This method has already been applied to the interaction between a single spinor field and a two-component neutrino field ([4]).

This paper presents a model in which the simplest class of diagrams (ladder diagrams for a one-particle state) are taken into account to see what relation between the various parameters of the σ-interaction emerges if the muon and electron are describable by a leptonic field, and also to examine whether or not the σ-meson can possibly exist.

The scalar interaction is discussed in two ways. In Section **2**, the scalar interaction between the muon and σ-meson, which gives rise to a negative self-mass, is calculated to study the possibility that the electron may be regarded as a light muon. In Section **3**, the electron and the muon are regarded as two eigenstates of the scalar interaction between a leptonic field and the σ-meson field. In both methods the requirement that the anomalous magnetic moment of the muon should be less than $10^{-3}\,(e/2M_\mu)$ is imposed. Some concluding remarks are made in Section **4**.

([1]) J. SCHWINGER: *Ann. Phys.*, **2**, 407 (1957).

([2]) M. GELL-MANN and F. LOW: *Phys. Rev.*, **84**, 350 (1951).

([3]) This one-particle wave function is closely related to that discussed in G. C. WICK: *Ann. Rev. Nucl. Sci.*, **8**, 1 (1958).

([4]) H. KITA and E. PREDAZZI: *Nuovo Cimento*, **17**, 908 (1960).

2. – Calculation of self-mass.

The self-mass of the muon, arising from the scalar σ-meson interaction, is given to the second order in the coupling constant as

$$(2)\qquad \delta m\bar{u}(0)u(0) = -\frac{ig^2}{(2\pi)^4}\,\bar{u}(0)\int_0^{\Lambda^2}\mathrm{d}L\int \mathrm{d}k\,\frac{i(p_\mu\gamma_\mu - k_\mu\gamma_\mu) - m}{(k-p)^2 + m^2}\,\frac{1}{(k^2+\mu^2+L)^2}\,u(0),$$

where the momentum of the muon is put equal to zero and a Feynman invariant cut-off function $\Lambda^2/(\Lambda^2+k^2)$ has been inserted under the integral ([5]). This expression is written for $\hbar = c = 1$, and μ is the mass of the σ-meson. For later convenience, eq. (2) with $p^2 = -M^2$ is written in the form

$$(3)\qquad \delta m = (g^2/16\pi^2)\,\bar{u}(0)[i\gamma_\mu p_\mu\beta - m\alpha]\,u(0)\,,$$

where

$$(4)\qquad \alpha = \alpha(\Lambda) - \alpha(0)\,,$$

$$(5)\qquad \alpha(\Lambda) = \frac{1}{2M^2}\{\Lambda^2+\mu^2+M^2-m^2-[(\Lambda^2+\mu^2-m^2-M^2)^2-4m^2M^2]^{\frac12}\}\cdot$$
$$\cdot\ln\left(\frac{\Lambda^2+\mu^2}{m^2}\right)+\frac{1}{M^2}[(\Lambda^2+\mu^2-m^2-M^2)^2-4m^2M^2]^{\frac12}\cdot$$
$$\cdot\ln\left\{\frac{\Lambda^2+\mu^2+m^2-M^2-[(\Lambda^2+\mu^2-m^2-M^2)^2-4m^2M^2]^{\frac12}}{2m^2}\right\},$$

and

$$(6)\qquad \beta = \beta(\Lambda) - \beta(0)\,,$$

$$(7)\qquad \beta(\Lambda) = \frac{\Lambda^2}{2M^2} +$$
$$+\left\{\frac{M^2+m^2}{2M^2}+\frac{(\Lambda^2+\mu^2-m^2-M^2)[(\Lambda^2+\mu^2-m^2-M^2)^2-4m^2M^2]^{\frac12}}{4M^4}-\right.$$
$$\left.-\frac{(\Lambda^2+\mu^2-m^2-M^2)^2}{4M^4}\right\}\ln\left(\frac{\Lambda^2+\mu^2}{m^2}\right)-$$
$$-\frac{(\Lambda^2+\mu^2-m^2-M^2)}{2M^4}\cdot[(\Lambda^2+\mu^2-m^2-M^2)^2-4m^2M^2]^{\frac12}\cdot$$
$$\cdot\ln\left\{\frac{\Lambda^2+\mu^2+m^2-M^2-[(\Lambda^2+\mu^2-m^2-M^2)^2-4m^2M^2]^{\frac12}}{2m^2}\right\}.$$

([5]) The criticism raised by S. Sunakawa and K. Tanaka: *Phys. Rev.*, **115**, 754 (1959) toward such an invariant cut-off function is not valid here, because the muon is not a strongly interacting system. The purpose of the cut-off here is to make an otherwise divergent integral finite.

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The requirement that the contribution to the anomalous muon magnetic moment due to the σ-interaction be less than $\delta F = 10^{-3}(e/2M_\mu)$ imposes a restriction on the values M_μ^2 and μ^2. This restriction is satisfied for large values of μ^2/M_μ^2 for which δF is given to the second order in the coupling constant as

$$\delta F = \frac{e}{2M_\mu}\frac{g^2}{8\pi^2}\frac{M_\mu^2}{\mu^2}\left(\ln\frac{\mu^2}{M_\mu^2} - \frac{5}{3}\right) \leqslant 10^{-3}\frac{e}{2M_\mu}\,. \tag{8}$$

In other words, specifying that $g^2/4\pi = 2$ requires $\mu^2/M_\mu^2 \geqslant 1800$.

Since $M_\mu^2 = m^2$ and $i\gamma_\mu p_\mu \to -m$ for the self-mass, substitution of eqs. (4) to (7) into (3) and keeping only the leading terms, leads to

$$\frac{\delta m}{m} = -\frac{g^2}{16\pi^2}(\alpha + \beta) \simeq -\frac{g^2}{16\pi^2}\left(\frac{3}{2}\right)\ln\left(\frac{\Lambda^2 + \mu^2}{\mu^2}\right). \tag{9}$$

The observed mass of the electron is given as

$$M_e = m + \delta m = 207 - 206 = 1\,,$$

so that eq. (9) is expressible as

$$\frac{g^2}{16\pi^2}\left(\frac{3}{2}\right)\ln\left(\frac{\Lambda^2 + \mu^2}{\mu^2}\right) = 206\,,$$

which leads to the value $\Lambda^2/\mu^2 \to \infty$ for $g^2/4\pi \leqslant 10$. We thus conclude that one cannot regard the electron as a light muon if the σ-interaction is responsible for the mass difference ([6]).

3. – Eigenvalue problem.

The consequence of regarding the muon and electron states as eigenstates of the leptonic field $\psi(x)$ will now be examined. For this purpose, the function $G(w-z)$ of eq. (1) is taken as

$$G(\omega - z) = (g^2/8)\,S_F(\omega - z)\,\Delta_F(\omega - z)\,. \tag{10}$$

([6]) One would expect that the cut-off is not too much larger than the masses that appear in the theory. For a different viewpoint see W. S. Cowland: *Nucl. Phys.*, 8, 397 (1958). In the present framework, the approximation

$$\ln[(\Lambda^2 + \mu^2)/\mu^2] \approx \ln(\Lambda^2/\mu^2) = -\ln(\mu^2/\Lambda^2)\,,$$

with $\mu^2/\Lambda^2 \approx 10^2$ has been taken in this reference.

Then the sum in eq. (1) includes only the simplest self-energy diagrams which, at no time have more than one σ-meson in the field. There is no rigorous support for this model and unfortunately the estimation of the remainder that appears in a relativistic field theory is formidable.

This model becomes more valid when the value of the coupling constant is smaller, as can be seen from the following argument. In a one-particle state, the particle emits and reabsorbs the σ-meson for a very long time. If the coupling constant is small, then the probability for finding one virtual σ-meson in the field is larger than that of finding two or more virtual σ-mesons simultaneously. The particle may successively emit and reabsorb any number of virtual σ-mesons which are taken care of in this model. The neglected diagrams, which correspond to the cases in which two or more σ-mesons exist simultaneously in the field, become unimportant if the coupling constant is small.

Now substitute eq. (10) into (1) and use the limiting procedure of Gell-Mann and Low to project out the one-particle states $\chi_\alpha(x) = \langle 0|\psi(x)|\alpha\rangle$ in the relation

$$-\tfrac{1}{2} S'_F(x-y) = \sum_n \langle 0|\psi(x)|n\rangle\langle n|\overline{\psi}(y)|0\rangle = \sum_n \chi_n(x)\overline{\chi}_n(y)\,, \qquad x_0 > y_0\,,$$

where the states $|n\rangle$ are eigenstates of the momentum four-vector. The result is

$$(i\gamma_\mu p_\mu + m)\chi_\alpha = A\chi_\alpha\,, \tag{11}$$

where

$$A = -(g^2/6\pi^2)(i\gamma_\mu p_\mu \beta - m\alpha)\,. \tag{12}$$

The right-hand side of eq. (11) should include another term arising from the interaction between the leptonic field and the electromagnetic field and having the sign opposite to that of the σ-meson term because its self-mass is positive. This term, however, can be neglected in comparison to the term A because the coupling constant of the electromagnetic interaction is only about a hundredth of that of the σ-interaction ([7]). Combining eq. (11) and (12) and the condition $\boldsymbol{p} = 0$, one gets

$$\left[-\gamma_0 M + m + \frac{g^2}{16\pi^2}(-\gamma_0 M\beta - m\alpha)\right]\chi_\alpha = 0\,. \tag{13}$$

([7]) Making $g^2/4\pi \lessdot 1$ is not acceptable because such values require excessively large cut-off values, of the order of 10^2 nucleon masses.

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The requirement that eq. (13) has a non-zero solution is

$$m\left(1-\frac{g^2}{16\pi^2}\,\alpha\right)=\pm\, M\left(1+\frac{g^2}{16\pi^2}\,\beta\right). \tag{14}$$

This result is an eigenvalue equation expressing M in terms of the parameters Λ, μ, m, and $g^2/4\pi$. The problem is to specify these parameters so that M_μ and M_e satisfy eq. (14). In addition, the following conditions must be satisfied in order to have a well defined problem. First, Λ should be the largest mass that appears in the theory. Second, m should be larger than M_μ and M_e so that the limiting procedure is valid. Third, δF should be less than $10^{-3}(e/2M_\mu)$ to avoid contradicting the experimental data on the anomalous magnetic moment of the muon, a value that is adequately accounted for by the electromagnetic interaction. When this condition is satisfied for the muon it is also satisfied for the electron. Forth, μ should be sufficiently massive so that the σ-interaction will be short range in order to accord with experimental data on muon scattering, etc.

For the numerical work, eq. (5) and (7) are not convenient because M, which will be one of the smaller masses (M_e or M_μ), appears in the denominator. Consequently α and β are expanded in terms of $M^2/\Lambda^2=x$ so that, to the first order in x, eq. (4) and (6) become

$$\begin{aligned}\alpha=&\frac{1+n}{1+n-\varkappa}\ln\left(\frac{1+n}{\varkappa}\right)-\frac{n}{\varkappa-n}\ln\left(\frac{\varkappa}{n}\right)+\left[\frac{n+\varkappa}{2(\varkappa-n)^2}-\frac{(1+n+\varkappa)}{2(1+n-\varkappa)^2}\right.\\&\left.+\frac{(1+n1)\varkappa}{(1+n-\varkappa)^3}\ln\left(\frac{1+n}{\varkappa}\right)-\frac{n\varkappa}{(\varkappa-n)^3}\ln\left(\frac{\varkappa}{n}\right)\right]x=\alpha_1+\alpha_2x\,,\end{aligned}\tag{15}$$

and

$$\begin{aligned}\beta=\frac{1}{2}\Biggl\{&\frac{\varkappa}{(1+n-k)(\varkappa-n)}+\ln\left(\frac{1+n}{\varkappa}\right)+\frac{(n^2-2n\varkappa)}{(\varkappa-n)^2}\ln\left(\frac{\varkappa}{n}\right)-\\&-\frac{\varkappa^2}{(1+n-\varkappa)^2}\ln\left(\frac{1+n}{\varkappa}\right)+\left[\frac{1}{(1+n-\varkappa)^3}\,\frac{2(1+n)^2-(1+n)\varkappa+5\varkappa^2}{3}+\right.\\&+\frac{1}{(\varkappa-n)^3}\,\frac{2n^2-n\varkappa+2\varkappa^2}{3}-\frac{n^2}{(\varkappa-n)^2(1+n-\varkappa)}-\\&\left.-\frac{2n\varkappa^2}{(\varkappa-n)^4}\ln\left(\frac{\varkappa}{n}\right)-\frac{2(1+n)\varkappa^2}{(1+n-\varkappa)^4}\ln\left(\frac{1+n}{\varkappa}\right)\right]x\Biggr\}=\beta_1+\beta_2x\,,\end{aligned}\tag{16}$$

where

$$n=\mu^2/\Lambda^2\qquad\text{and}\qquad\varkappa=m^2/\Lambda^2\,.$$

The accuracy of this expansion has been estimated and it is good for α, but the x^2 term is needed for a reliable result for β.

Substituting eq. (15) and (16) into eq. (14), leads to an equation for x which should be satisfied by $x = M_\mu^2/\Lambda^2$ and $x = M_e^2/\Lambda^2$, namely,

$$(17) \qquad (\varkappa)^{\frac{1}{2}}\left[1 - \frac{g^2}{16\pi^2}(\alpha_1 + \alpha_2 x)\right] = \pm\,(x)^{\frac{1}{2}}\left[1 + \frac{g^2}{16\pi^2}(\beta_1 + \beta_2 x)\right].$$

The values of n and $\varkappa$ are expected to be about $10^3 \div 10^4$ times as large as M_μ^2/Λ^2 so that M_e^2/Λ^2 can be considered as equal to zero. When $x = 0$ in eq. (17), the relation reduces to

$$(18) \qquad \alpha_1 = \frac{16\pi^2}{g^2}\,.$$

When this is substituted in eq. (17), the result is

$$(19) \qquad \alpha_2(\varkappa x)^{\frac{1}{2}} = \pm\,[(16\pi^2/g^2) + \beta_1 + \beta_2 x]$$

which should be satisfied by $x = M_\mu^2/\Lambda^2$.

The condition (8) can be used to obtain a relation between x and n once $g^2/4\pi$ is fixed. The method is to fix the value of $g^2/4\pi$ and adjust $\varkappa$ and n so that eq. (8), (18) and (19) are satisfied in addition to the four requirements mentioned in this section. The numerical values of Λ, m and μ that satisfy all the requirements are given in Table I in units of the mass of a nucleon. The masses M_μ and M_e are 0.1125 and 0, respectively and the value of δF is kept at approximately $10^{-3}\, e/2M_\mu$ for all values of $g^2/4\pi$.

TABLE I.

$g^2/4\pi$	2.01	3.92	9.90
Λ	183	56	32
m	4.96	7.13	13.2
μ	4.88	7.03	13.1

Values of x which are negative have been dropped. We are only interested in those value of x which lead to the observed masses of the electron and the muon.

The case of a pseudoscalar and vector σ-meson cannot be treated similarly because the self-mass to the second order in the coupling constant is positive, so that one cannot obtain the eigenvalue equation corresponding to eq. (11). Changing the value of δF, leads to another set of values of Λ, m and μ but the trend of the values remains the same.

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4. – Remarks.

It has been shown that the electron and the muon states are describable by a leptonic field which interacts with a σ-meson in a model in which the ladder diagrams are summed and a large cut-off is taken. It has not been possible to obtain a set of values in which the coupling constant, cut-off Λ, and σ-meson mass would simultaneously be small. If one wishes to obtain a σ mass of the order of a few nucleon masses, then the cut-off must be taken at about 10^2 nucleon masses.

The ratio Λ/μ of the cut-off Λ to the mass μ of the σ-meson depends sensitively on the scalar coupling parameter $g^2/4\pi$, because a summation over all the ladder diagrams has been carried out. If $g^2/4\pi$ is smaller, the probability of finding a virtual σ-meson in the field is smaller so that a larger cut-off Λ is required to obtain the same effect. This explains the dependence of Λ/μ on $g^2/4\pi$ in Table I. The near equality of the masses of the σ-meson and the leptonic field is accidental (8).

It is not without comfort that a model can be found, however unrealistic, that satisfies all the requirements mentioned in Section 3. Such a model is useful only if it provokes further thoughts on the possible explanation of the difference in mass between the muon and electron.

* * *

The author is very grateful to his colleagues for their helpful discussions and to Professor E. R. CAIANIELLO for his warm hospitality at the Istituto di Fisica Teorica.

(8) The mathematical reason is to enhance the terms α_2, β_1, and β_2 so as to obtain small values of x.

RIASSUNTO

La self-mass che si origina dall'interazione di un campo leptonico con un campo mesonico scalare neutro σ è calcolata con la teoria perturbativa, in modo da esaminare la possibilità che l'elettrone possa essere riguardato come un muone leggero, poichè il segno delle self-mass è negativo. Inoltre sono studiate le conseguenze di un modello nel quale un campo leptonico, interagendo con il mesone σ può descrivere l'elettrone e il muone come autostati.

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Search for Leptonic and Radiative Decays of Charged Σ-Hyperons.

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(ricevuto il 19 Maggio 1961)

Summary. — No leptonic decays have been observed in a large sample of charged Σ-hyperons (about 600 $\Sigma^{\pm}\to\pi^{\pm}+n$). Taking into account the efficiency of the method which has been used for the analysis of the events, the ratio $(e^{+}+\nu+n)/(\pi^{+}+n)$ is less than 1%, in accord with other authors. No $\Sigma^{+}\to\pi^{+}+\gamma+n$ events, with π^{+} of energy less than 75 MeV, have been detected. Up to now only two examples have been reported in the literature. Then the ratio $(\pi^{+}+\gamma+n)/(\pi^{+}+n)$ lies probably between 1 and 2%, consistently with the theoretical predictions.

In five stacks of photographic emulsions exposed to low energy K-beams, produced by the Bevatron, about 30 000 K-meson captures were observed, (the major part by the Bologna, München, Paris, Parma collaboration [1]).

From the analysis of a great part of these captures, about 600 charged Σ-hyperons were found which had a neutral baryon among the decay products. The charged secondaries of these hyperons have been examined in order to

(1) W. Alles, N. N. Biswas, M. Ceccarelli, m. Gessaroli, G. Quareni, M. Göing, K. Gottstein, W. Püschel, J. Tietge, G. T. Zorn, J. Crussard, J. Hennessy, G. Dascola and S. Mora: *Nuovo Cimento*, **11**, 771 (1959).

detect possible cases of leptonic decay, or radiative decay, in particular processes of the type: $e+\nu+n$, $e+\nu+\Lambda^0$, $\pi+\gamma+n$; which should be present together with the main decay mode $\pi+n$. Using the emulsion technique, it is, in principle, always possible to distinguish the electrons from the pions because of the bremsstrahlung of the electrons and the change of the ionization of the pions. Of course, this procedure is possible only if the tracks are sufficiently long, *i.e.* some centimeters in our case.

Because of the background due to the numerous light mesons, which accompagny the K^--beam, it was rather difficult to follow the tracks through the stack unambiguously. Therefore, only few secondary tracks were analysed according to the said procedure. For the major part of the events, the distinction between the different decay processes was based on the experimental separation of a two-body decay from a three-body decay. This was obtained by means of multiple scattering measurements, *i.e.* by comparing the obtained results with the expected ones.

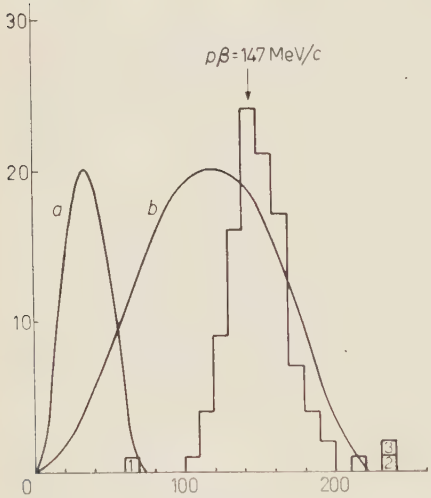

Fig. 1. – $p\beta$ distribution of 109 secondaries of Σ-hyperons decayed at rest. The curves *a* and *b* represent the electron spectra for the processes $e^++\nu+\Lambda^0$ and $e^++\nu+n$ respectively.

Each secondary track has been carefully followed in the emulsion in which it was generated, and in the next emulsions as long it was possible without ambiguity.

Provided the dip angle was smaller than 30° and the track sufficiently long, measurements of $p\beta$ were carried out.

As for the tracks longer than 1 mm per plate the obtained $p\beta$ values can be regarded as quite reliable, while for shorter tracks the obtained values must be considered, in the majority of the cases, as lower limits only. The main difficulty of this kind of measurements arises from the gelatine distortion, the effects of which can be corrected only when the available tracks are long enough.

The $p\beta$ distribution for the secondary particles which belong to the above said selection, is reported in Fig. 1; it refers to 109 decays at rest which we have consequently classified as Σ^+'s.

The spectra of the electrons, calculated by assuming they depend on the phase-space only, are also reported in Fig. 1 in arbitrary units.

In Fig. 2 are reported the momenta in the center-of-mass system of the secondary particles which arise from 51 Σ's in flight. The average mass

between the Σ^+ and the Σ^- has been used for the required kinematical calculations.

Both Fig. 1 and 2 show that the results of the measurement are spread around the expected value for the secondary π-meson of the two-body decay.

Three measurements (no. 1, 2 and 3 in Fig. 1) have rather unexpected values of $p\beta$.

The secondary track number 1 stops in flight in the emulsion after 5 mm; its grain density being about twice the minimum, the particle has been identified as a π-meson of 36 MeV. The pion associated with the K^- capture is positive; hence the hyperon must be negative.

This event could be interpreted as a radiative decay of a very slow Σ^-, but we prefer to explain it as a capture of a Σ^- producing a rather heavy hyperfragment which undergoes mesonic decay. In fact, the Σ^- seems to be really at rest and a final blob could account for the nuclear recoil.

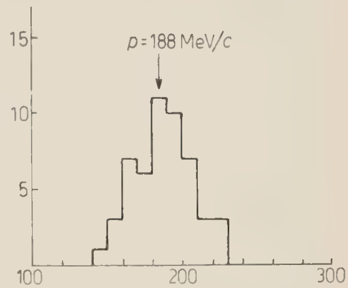

Fig. 2. – Distribution of the momenta in the c.m. system for 51 secondaries of Σ-hyperons decayed in flight.

The $p\beta$'s of the events number 2 and 3 are exceptionally high, but the results of the ionization measurements are closely in accord with the mean value we have obtained by measuring many secondaries of Σ's at rest, and not with the plateau ionization as one would expect for fast electrons. We wish to underline, in particular, one of the two events; here, the secondary tracks stops in the emulsion after 13 mm and its multiple scattering gives $p\beta = (234 \pm 19)$ MeV/c, which would be compatible with the electron spectrum. However it is very hard to support the presence of a positron, because of the ionization which is significantly higher than those of the fast electrons. The measured electrons (from μ-mesons or from energetic γ-rays) show a fairly constant ionization, so that even a slow increase with the energy seems to be excluded.

Apart from these three events, the experimental distribution is in accord with the assumption that all the measured secondaries are π-mesons of the two-body decay; the width of the distribution is consistent with the errors we have calculated for each event. Slow electrons under 100 MeV would have been detected, if present in the selected samples.

Also grain counts were made for about one third of the events at rest; the results were always in accord with the ionization expected for the mono-energetic secondary pion and never for an electron.

Another group of 37 secondaries, which do not satisfy the selection criteria has been also measured and no $p\beta$'s smaller than 80 MeV were recorded. Besides, 15 tracks have been followed until the particles come to rest, or long

enough to establish, with the help of ionization measurements, their mesonic nature.

Therefore we conclude that no evidence has been found for any kind of leptonic decay. The ratio $(e+\nu+\Lambda^0)/(\pi+n)$ is smaller than 1/196. Provided the electron energy spectrum for the $e+\nu+n$ process is not very different from that we have assumed, the ratio $(e+\nu+n)/(\pi+n)$ becomes smaller than 1%. Because the number of Σ^-'s present in our sample is very small, the conclusion we have drawn refers to the Σ^+ decay.

We summarize here briefly the present situation of the leptonic decay of the Σ-hyperons. LEITNER *et al.* (2) investigated the same problem by means of a Hydrogen Bubble Chamber and they, too, gave negative results. Positive evidence was given by HORNBOSTEL and SALANT (3) who report an event, found in emulsion, which is interpreted as a leptonic decay of a Σ of undetermined sign, and by FRANZINI and STEINBERGER (4) who report an example of leptonic decay of a Σ^- detected by means of a propane chamber.

Theoretically, the ratio $(e+\nu+\Lambda^0)/(\pi+n)$ is expected to be of the order of 10^{-4} and so far the experiments are not statistically sufficient to check such a small value.

By means of the universal Fermi interaction theory one can predict a ratio of a few percent for the $e+\nu+n$ process. For this last case different assumptions lead to the forbiddance of alternatively, the leptonic decay of the Σ^+ (5) or of the Σ^- (6). The whole experimental results indicate that the ratio $(e+\nu+n)/(\pi+n)$ is significantly lower than the predicted one.

In our sample of hyperons no evidence has been found for the radiative decay. Taking into account the results of the multiple scattering and of the ionization measurements for the decays at rest, positive pions of energy less than 75 MeV can be excluded.

Up to now, two possible events $\Sigma^+ \to \pi^+ + \gamma + n$ have been reported in the literature (7,8). They were found in small samples of analysed hyperons. Thus the experimental ratio $(\pi^+ + \gamma + n)/(\pi^+ + n)$ lies between 1 and 2 per cent. Such order of magnitude agrees with that calculated by BARSHAY and BEHRENDS (9).

(2) J. LEITNER, P. NORDIN, A. H. ROSENFELD, F. T. SOLMITZ and R. D. TRIPP: *Phys. Rev. Lett.*, **3**, 186 (1959).

(3) J. HORNBOSTEL and E. O. SALANT: *Phys. Rev.*, **102**, 502 (1956).

(4) P. FRANZINI and J. STEINBERGER: *Phys. Rev. Lett.*, **6**, 281 (1961).

(5) R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958).

(6) R. P. FEYNMAN: unpublished.

(7) E. M. FRIEDLÄNDER: *Phys. Rev. Lett.*, **4**, 528 (1960).

(8) C. M. GARELLI, B. QUASSIATI and M. VIGONE: *Nuovo Cimento*, **16**, 960 (1960).

(9) S. BARSHAY and R. E. BEHRENDS: *Phys. Rev.*, **114**, 931 (1959).

* * *

We wish to thank the authors of the Bologna, München, Paris, Parma Collaboration for having put their scanning data at our disposal, Prof. M. CECCARELLI and Dr. F. SELLERI who helped us at the beginning of the work, Miss B. FRENTZEL-BEYME, Miss C. SOAVE, Mr. BERNO and Mr. DAINESE for their valuable contribution to the measurements.

The authors of the I.F.U.B. thank the Max-Planck-Institute for the kind hospitality.

RIASSUNTO

Sono stati cercati possibili casi di decadimento leptonico di iperoni Σ carichi, in un gruppo di ~ 600 iperoni osservati in emulsione nucleare e precedentemente classificati come $\Sigma^{\pm} \to \pi^{\pm} + n$. Non è stato trovato nessun decadimento leptonico. Tenuto conto dell'efficienza del metodo impiegato per l'analisi degli eventi il rapporto $(e^+ + \nu + n)/(\pi^+ + n)$ risulta inferiore all'1%, in accordo con i risultati ottenuti da altri autori. Non è stato osservato nessun evento del tipo $\Sigma^+ \to \pi^+ + \gamma + n$, con π^+ di energia minore di 75 MeV. Due di questi eventi sono stati finora trovati in altri laboratori. Perciò il rapporto $(\pi^+ + \gamma + n)/(\pi^+ + n)$ è probabilmente compreso tra 1 e 2% in accordo, come ordine di grandezza, con le previsioni teoriche.

Gauge Covariance of Spinor Geometry (*).

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(ricevuto il 28 Dicembre 1960)

It is often stated [1,2] that the vanishing of the covariant derivative of the alternating spinor [3] ε_{mn} is incompatible with the gauge invariance of spinor geometry. It was however shown by BERGMANN [4] that $\varepsilon_{mn;\mu}$ vanishes *identically* if account is taken of the density character of ε_{mn}. The purpose of this note is to generalize some of Bergmann's results.

Let $T_{m'}{}^{n}$ be the transformation matrix between two spinor frames, and let $T = \mathrm{Det}(T_{m'}{}^{n})$. Following CORSON [5], we define a covariant *pseudospinor* ψ_m of weight p and antiweight q by the transformation law:

$$\psi_{m'} = T^{q}\,\bar{T}^{q}\,T_{m'}{}^{n}\,\psi_n\,,$$

with similar laws for contravariant (and higher order) pseudospinors. Notice that $\psi_{\dot{m}} = \bar{\psi}_m$ has weight q and antiweight p.

For instance ε^{mn} has weight 1 and antiweight 0, and it follows from

$$g^{\mu\dot{m}n}\,g_{\mu}{}^{\dot{r}s} = 2\varepsilon^{\dot{m}\dot{r}}\varepsilon^{ns}\,,$$

that the weight and antiweight of $g^{\mu\dot{m}n}$ are both equal to $\frac{1}{2}$.

Now, the Dirac equations [6]

$$g^{\mu\dot{m}n}\psi_{n;\mu} = -k\varepsilon^{\dot{m}\dot{n}}\varphi_{\dot{n}}\,, \qquad g^{\mu\dot{m}n}\varphi_{\dot{m};\mu} = k\varepsilon^{mn}\psi_m\,,$$

(*) This work was partly supported by the U. S. Air Force through the European Office of the Air Research and Development Command.

(1) W. L. BADE and H. JEHLE: *Rev. Mod. Phys.*, **25**, 714 (1953). This paper contains numerous references to earlier literature.

(2) H. A. BUCHDAHL: *Nuovo Cimento*, **11**, 496 (1959).

(3) Latin indices label spinor components, Greek indices, vector components. A comma denotes partial differentiation, a semicolon, covariant differentiation.

(4) P. G. BERGMANN: *Phys. Rev.*, **107**, 624 (1957).

(5) E. M. CORSON: *Introduction to Tensors, Spinors and Relativistic Wave-Equations* (London, 1953), pp. 14-16.

(6) E. M. CORSON: *Introduction to Tensors, Spinors and Relativistic Wave-Equations* (London, 1953), pp. 104-105.

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imply that if $\varphi_{\dot{n}}$ has weight u and antiweight v, then ψ_n has weight $u-\frac{1}{2}$ and antiweight $v+\frac{1}{2}$. Furthermore, since the Lagrangian [6]

$$L = \mathrm{Im}\,(\psi_m g^{\mu\dot{n}m}\psi_{\dot{n};\mu} + \varphi_{\dot{m}} g^{\mu\dot{m}n}\varphi_{n;\mu} + 2k\psi_m \varepsilon^{mn}\varphi_n)\,,$$

must be invariant under spinor transformations, then it follows that $u+v=-\frac{1}{2}$.

Now, the covariant derivatives of our pseudospinors are

$$\psi_{n;\mu} = \psi_{n,\mu} - \psi_m \Gamma^m{}_{n\mu} - (u-\tfrac{1}{2})\,\psi_n \Gamma^m{}_{m\mu} - (v+\tfrac{1}{2})\,\psi_n \Gamma^{\dot{m}}{}_{\dot{m}\mu}\,,$$

$$\varphi_{\dot{n};\mu} = \varphi_{\dot{n},\mu} - \varphi_{\dot{m}}\Gamma^{\dot{m}}{}_{\dot{n}\mu} - u\,\varphi_{\dot{n}}\Gamma^m{}_{m\mu} - v\,\varphi_{\dot{n}}\Gamma^{\dot{m}}{}_{\dot{m}\mu}\,.$$

Taking, for the sake of simplicity, Cartesian coordinates (the generalization to curvilinear coordinates is trivial), we have $\Gamma^m{}_{n\mu}=i\delta^m{}_n a_\mu$, where a_μ is some vector. By virtue of $u+v=-\frac{1}{2}$, we thus obtain

$$\psi_{n;\mu} = \psi_{n,\mu} - 2iu(a_\mu + \bar{a}_\mu)\psi_n\,, \qquad \varphi_{\dot{n};\mu} = \varphi_{\dot{n},\mu} - 2iu(a_\mu + \bar{a}_\mu)\,\varphi_{\dot{n}}\,.$$

We see that *only the real part of* a_μ *is relevant.* The correspondence with quantum mechanics implies

$$e\hbar A_\mu/c = 2u(a_\mu + \bar{a}_\mu)\,.$$

Particles of different charges correspond to pseudospinors of different weights.

An important implication of the above theory is that a_μ is only partly determined by the electromagnetic potential: its real part is fixed only up to the (hitherto unknown) multiplicative constant u, while its imaginary part is arbitrary and perhaps devoid of physical interest. The consequences of the above facts on a unified theory of gravitation, spin and electromagnetism will be discussed elsewhere.

A Simple Derivation of the Geodesic Equations of Motion from the Matter Tensor in General Relativity Using the δ-Function.

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(ricevuto il 4 Marzo 1961)

Einstein's theory of gravitation rests on two fundamental equations:

— Field equations:

$$R_{\mu\lambda} - \tfrac{1}{2} g_{\mu\lambda} R = -k T_{\mu\lambda}\,; \qquad T^{\mu\lambda}{}_{;\lambda} = 0\,. \tag{1}$$

— Geodesic equations of motion:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \begin{Bmatrix} \mu \\ \alpha\ \beta \end{Bmatrix} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0\,. \tag{2}$$

Explicit equations of motion of a system of particles from the field equations alone have been obtained by

(i) Einstein, Infeld, Hoffmann (1), without using the matter tensor $T_{\mu\lambda}$ and considering the particles as point-singularities.

(ii) Fock (2) and his school, using the matter tensor and a harmonic coordinate system.

(iii) Infeld (3), who combines the two approaches by using the δ-function. Simple derivations of eq. (2) from (1) have been given by various authors, namely

(*a*) Eddington (4);

(*b*) Fock (5), in two different ways using the continuity equation;

(*c*) Jordan (6) using the δ-function.

(1) A. Einstein, L. Infeld and B. Hoffmann: *Ann. Math.*, **41**, 455 (1940).
(2) V. A. Fock: *Žurn. Èksp. Teor. Fiz.*, **1**, 81 (1939), see also (5).
(3) L. Infeld: *Acta Phys. Pol.*, **13**, 187 (1954).
(4) A. S. Eddington: *Mathematical Theory of Relativity* (Cambridge, 1954), p. 126.
(5) V. A. Fock: *Theory of Space Time and Gravitation* (London, 1959), p. 215.
(6) P. Jordan: *Schwerkraft und Weltall* (Braunschweig, 1955), p. 82.

We give here a simple derivation of eq. (2) from (1) which is very similar to that of Jordan's except that we use also the equation of continuity.

We first consider a continuous distribution of matter (perfect fluid) with density ϱ. Thus we have

$$T^{\mu\lambda} = \varrho \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s}; \qquad T^{\mu\lambda}{}_{;\lambda} = 0 . \tag{3}$$

The equation of continuity can be written as

$$\left(\varrho \frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\right)_{;\lambda} = 0 \tag{4}$$

We now go over from a continuous distribution of matter to a single particle of unit mass whose path is given by $\bar{x}^\mu = \bar{x}^\mu(s)$. Then we can write

$$\varrho = \frac{\delta(x-\bar{x})}{\sqrt{-g}} \qquad \text{where} \qquad \delta(x-\bar{x}) = \prod_\mu \delta(x^\mu - \bar{x}^\mu) . \tag{5}$$

The factor $\sqrt{-g}$ is introduced in order that ϱ should be a scalar. We note that $\delta(x-\bar{x})$ *transforms like a scalar density*, because from its definition, for a scalar function $f(x)$ we must have

$$\int \delta(x-\bar{x}) f(x)\, \mathrm{d}x = f(\bar{x}) = \text{scalar} .$$

So for a single particle eq. (3) and (4) become

$$\left(\frac{\delta(x-\bar{x})}{\sqrt{-g}} \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\right)_{;\lambda} = 0 , \tag{6}$$

$$\left(\frac{\delta(x-\bar{x})}{\sqrt{-g}} \frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\right)_{;\lambda} = 0 . \tag{7}$$

Eq. (7) is to be considered as the equation of continuity for a single particle. Eq. (6) can be written as

$$\frac{\delta(x-\bar{x})}{\sqrt{-g}} \frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right)_{;\lambda} + \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\left(\frac{\delta(x-\bar{x})}{\sqrt{-g}} \frac{\mathrm{d}x^\lambda}{\mathrm{d}s}\right)_{;\lambda} = 0 .$$

The second term vanishes in view of (7) and the first term gives

$$\frac{\delta(x-\bar{x})}{\sqrt{-g}} \left\{\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \left\{ {\mu \atop \alpha\,\beta} \right\} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\right\} = 0 ,$$

which implies

$$\frac{\mathrm{d}^2\bar{x}^\mu}{\mathrm{d}s^2} + \left\{ {\mu \atop \alpha\,\beta} \right\} \frac{\mathrm{d}\bar{x}^\alpha}{\mathrm{d}s}\frac{\mathrm{d}\bar{x}^\beta}{\mathrm{d}s} = 0 ,$$

i.e. the geodesic equation of motion.

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The Contribution of the Born Terms to Photoproduction of Pions at High Energies.

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(ricevuto l'8 Marzo 1961)

The phenomenological analysis of the reactions

$$\gamma+\mathrm{p}\begin{array}{l}\nearrow \mathrm{n}+\pi^+\,,\\ \searrow \mathrm{p}+\pi^0\,,\end{array}\tag{1}$$

at high energies has been discussed by several authors [1-4], assuming that the production amplitude can be approximated in the following way [2]:

$$F^{\pi^+}=F^{\pi^+}_{\mathrm{B,e}}+F^{\pi^+}_1+F^{\pi^+}_2+F^{\pi^+}_3\,,\tag{2}$$

$$F^{\pi^0}=F^{\pi^0}_1+F^{\pi^0}_2+F^{\pi^0}_3\,.\tag{3}$$

F_1, F_2, F_3 are pure multipole transitions leading to the excited states of the nucleon which belong to the first, second and third resonance. The « electric Born term » $F^{\pi^+}_{\mathrm{B,e}}$ is the amplitude following from relativistic perturbation theory in 2-nd order, if the proton is treated as a Dirac particle (without anomalous magnetic moment). It is the aim of this note to point out that in (2) and (3) further terms should be added, which are expected to contribute not less than the resonance terms.

The dispersion relations for photoproduction [5] suggest that in addition to $F^{\pi^+}_{\mathrm{B,e}}$ the corresponding term for π^0-production and the « magnetic Born terms » $F_{\mathrm{B,m}}$ are essential parts of the production amplitude. The magnetic Born terms are equal to the result of 2-nd order perturbation theory (PS, PS), if the interaction of the anomalous magnetic moment of the nucleon with the electromagnetic field is considered. We have evaluated the cross sections calculated from the total Born

(1) R. R. Wilson: *Phys. Rev.*, **110**, 1212 (1958).
(2) R. F. Peierls: *Phys. Rev. Lett.*, **1**, 174 (1958); *Phys. Rev.*, **118**, 325 (1960).
(3) A. M. Wetherell: *Phys. Rev.*, **115**, 1722 (1959).
(4) L. F. Landovitz and L. Marshall: *Phys. Rev. Lett.*, **3**, 190 (1959).
(5) G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu: *Phys. Rev.*, **106**, 1345 (1957).

term [6] in the c.m. system:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{\pi^0}_{\mathrm{Born}} = e^2 f^2 \left(\frac{W}{M}\right)^2 \frac{1}{E_2 + q\cos\theta}\,\frac{q}{kW}\left\{(1+g'_{\mathrm{p}})^2(\omega_q - q\cos\theta) + \right.$$
$$\left. + \frac{q^2}{2}\sin^2\theta\left[g'^2_{\mathrm{p}}\left(\frac{M}{W}\right)^2 - \frac{W}{k^2(E_2+q\cos\theta)^2}\right]\right\}, \tag{4}$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)^{\pi^+}_{\mathrm{Born}} = 2e^2 f^2 \left(\frac{M}{W}\right)^2 \frac{q}{k}\left\{1 - (g_{\mathrm{p}} + g_{\mathrm{n}})\frac{\omega_q - q\cos\theta}{W} - \frac{q^2}{2k^2}\,\frac{\sin^2\theta}{(\omega_q - q\cos\theta)^2} + \right.$$
$$\left. + \frac{1}{4W(E_2+q\cos\theta)}\left[(g'_{\mathrm{p}}+g_{\mathrm{n}})^2(\omega_q - q\cos\theta)^2 + (g'^2_{\mathrm{p}} + g^2_{\mathrm{n}})\frac{q^2W^2}{M^2}\sin^2\theta\right]\right\}. \tag{5}$$

The notation is the same as in [5]. Figs. 1 and 2 show that there are large differences between the calculated Born cross sections and the experimental data which cannot be accounted for only by the resonance terms.

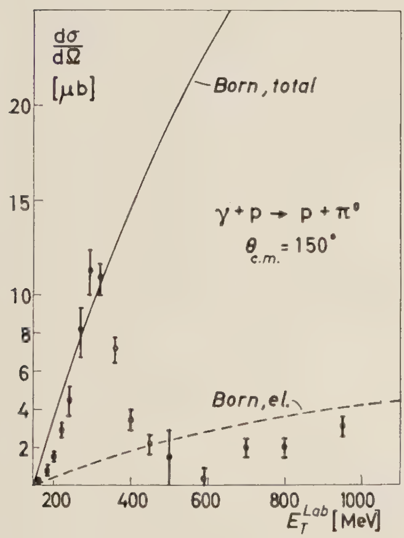

Fig. 1. – Born cross sections and experimental values for π^0-production at 150°. « Born total » corresponds to eq. (4), « Born el. » to the same equation and $g'_{\mathrm{p}} = 0$; $f^2 = 0.080$.

In the case of π^0-production at low energies, it is known that there exists a large non-resonant effect of the final state interaction which partially compensates the magnetic Born term [5,7]. Even if the compensation is almost complete at high energies, it is not sure that the rest is small compared with F_2 because of the magnitude of $F_{\mathrm{B,m}}$.

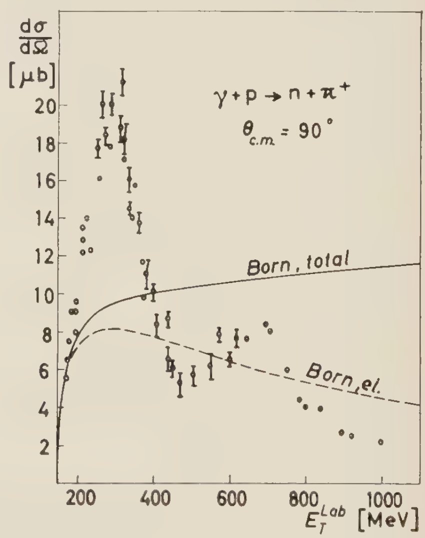

Fig. 2. – Born cross sections and experimental values for π^+ production at 90°, calculated from (5). $f^2 = 0.080$.

[6] M. F. KAPLON: *Phys. Rev.*, **83**, 712 (1951).
[7] K. DIETZ, G. HÖHLER and A. MÜLLENSIEFEN: *Zeits. Phys.*, **159**, 77 (1960).

In order to demonstrate that the same term is responsible for the compensation at higher energies, we have calculated the angular distribution for $\gamma+p \to p+\pi^0$ at 500 MeV, using the following ansatz:

$$F^{\pi^0} = F^{\pi^0}_{B,e} + F^{\pi^0}_{B,m} + F^{\pi^0}_{1} + F' . \tag{6}$$

The amplitude of the first resonance was taken from the paper of CHEW *et al.* (5), modified by subtracting the Born part of this multipole (8). As value of α_{33} the result of the phase shift analysis was taken. F' is the large dispersion integral contribution, estimated from the expression for E^{res}_{0+} which was used in (7). The angular distribution following from (6) fits fairly well the experimental values, in particular at large angles, where the Born cross section is too high by a factor 10 (Fig. 1). Taking into account an imaginary part of E_{0+} gives only a small correction, if it is estimated in the usual way.

The Born term has an important influence on the polarization of the outgoing nucleon. We have calculated the complex $M\,1\,P_{\frac{3}{2}}$-amplitude of Wilson's model under the condition that this amplitude together with $F^{\pi^0}_{B,e}$ fits the experimental values for π^0-production at 690 MeV. The polarization at 90° is $+0.4$. Therefore the simple argument of SAKURAI (9) has to be replaced by a more detailed investigation. This result is related to the letter of PELLEGRINI and STOPPINI (10).

In the case of π^+-production the magnetic Born term is smaller, but there seems to exist a similar compensating effect as discussed above for π^0, because $F^{\pi^+}_{B,e}$ alone

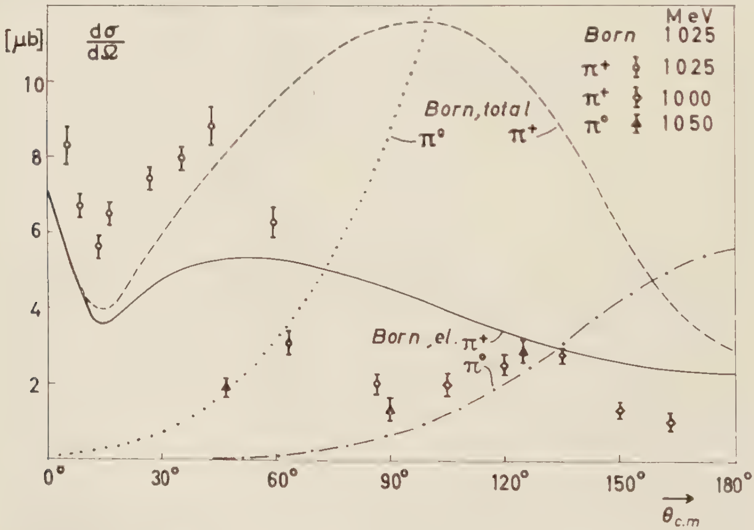

Fig. 3. – Angular distribution at 1025 MeV. Experimental values: BOYDEN and WALKER (11), DIXON and WALKER (12), JACKSON, DE WIRE and LITTAUER (13). $f^2 = 0.080$.

(8) B. F. TOUSCHEK: *Suppl. Nuovo Cimento*, **14**, 278 (1959).
(9) J. J. SAKURAI: *Phys. Rev. Lett.*, **1**, 258 (1958).
(10) C. PELLEGRINI and G. STOPPINI: *Nuovo Cimento*, **17**, 269 (1960).
(11) R. L. WALKER: *Proc. of the Rochester Conf.* (1960). We are indebted to Prof. WALKER for sending us a preprint.
(12) F. P. DIXON and R. L. WALKER: *Phys. Rev. Lett.*, **1**, 458 (1958).
(13) H. E. JACKSON, J. W. DE WIRE and R. M. LITTAUER: *Phys. Rev.*, **119**, 1381 (1960).

represents the general behaviour of the non-resonant part of the amplitude (Fig. 2). The angular distribution at 1025 MeV is shown in Fig. 3.

It is well known that the expansion: $A+B\cos\theta+\dots$ of the cross section cannot be used for π^+-production, because of the denominator $(1-v_\pi\cos\theta)$ in one of the Born terms. For π^0-production the denominator $(1+v_N\cos\theta)$ occurs. Since the velocity of the outgoing proton is not small at high energies ($v_N=0.25$ at 400 MeV, $v_N=0.38$ at 600 MeV) the expansion is somewhat doubtful in this case too, particularly for the small coefficient B.

* * *

The authors wish to thank Mr. VON SCHLIPPE and Mr. ZWINGENBERGER for assistance with the calculations.

A Model for K^0/K^+ Branching Ratios for Σ Production in High Energy π^--p Collisions.

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(ricevuto il 10 Marzo 1961)

1. – Introduction.

Recently, the reaction

$$\pi^- + p \to \Sigma^\pm + K + n\cdot\pi\,,$$

has been studied at laboratory pion momentum 7 GeV/c [1] and 16 GeV/c [2]. A considerable backward peaking of the Σ (in the c.m.s.) was found, and in the 16 GeV/c experiment the Σ^+/Σ^- ratio was found to be $1.8^{+0.7}_{-0.5}$. One may say that in these reactions the baryon tends to conserve not only its direction of motion, but also its charge.

These two features can be explained in terms of graphs, in which only a few boson lines are connected to the baryon line, and the multiple boson production is caused by the interaction of the incoming pion with one of the bosons coming from the baryon.

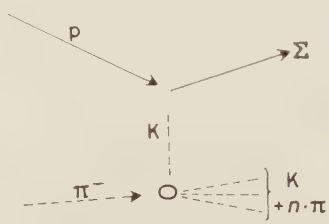
In this letter we show that by measuring the K^0/K^+ ratio for Σ^- production,

$$r_- = \frac{\sigma(\pi^- + p \to \Sigma^- + K^0 + n\cdot\pi)}{\sigma(\pi^- + p \to \Sigma^- + K^- + n\cdot\pi)}\,,$$

one can decide whether the $\pi\pi$ or the πK interaction is the important one. Moreover, for the πK interaction, a specific model is suggested (Section 3), and predictions are obtained not only for r_-, but also for r_+ and r_0 (the corresponding K^0/K^+ ratios for Σ^+ and Σ^0, Λ^0 production). The possible influence of a $K\pi$ resonance interaction is mentioned.

2. – Single-virtual-boson graphs.

Consider first graph 1. Here only one boson line is connected to the baryon



Graph 1.

(*) Work performed under US. Air Force contract nr. AF61(052)-47.

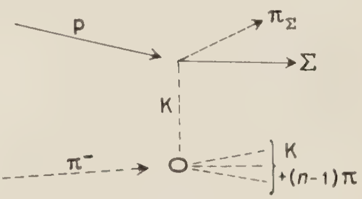
(1) M. I. SOLOVIEV: *Proceedings of the Annual International Conference on High Energy Physics at Rochester* (New York, 1960), p. 388.

line. From conservation of isospin, this graph will give (2/3) Σ^+, (1/3) Σ^0, and no Σ^-. The lower « vertex », which we shall call an mp vertex (multiple production vertex), represents the interaction of the incoming pion with the virtual kaon, leading to the creation of one kaon and n pions. For the charge distribution of the particles emerging from the mp vertex, we employ statistical theory, including isospin conservation. However, we do not calculate the mean value of n, but estimate it from the prong distribution found in (2) to be $\langle n\rangle = 4.3$. (Thus, our results refer to 16 GeV/c pion momentum, but they should be rather insensitive to pion momentum.) The charge distribution for the emerging kaon is then calculated for $n=4$, $n=5$ and $n=3$, and averaged with weights 4, 2 and 1, respectively. The following numbers are the obtained average values, their error should not exceed 15%. The statistical theory factors are taken from (3).

In this way we find that graph 1 gives 61% K^0 and 39% K^+ for Σ^+ creation, and 54% K^0 and 46% K^+ for Σ^0 creation.

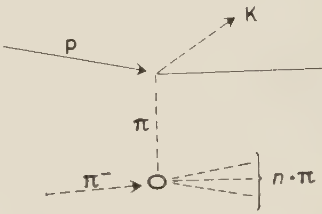
The momentum and angular distributions of the Σ's may be calculated according to a model of Chew and Low (4), which has been applied to pion-nucleon collisions (5). It gives the backward peak, although weaker than the experimental peak. (Here the kaon has been assumed to be pseudoscalar. It should be noted that assuming a scalar kaon improves the fit.)

For two bosons connected to the baryon, line there exist four different graphs. Consider first the two graphs, in which only one of the two bosons is virtual (graphs 2 and 3). Graphs



Graph 2.

of this kind have been recently discussed (6). Let us add the momentum of the other, real boson to the Σ momen-



Graph 3.

tum $\boldsymbol{p}_\Sigma$ to give a momentum $\boldsymbol{p}_{\Sigma'}$. Then the $\boldsymbol{p}_{\Sigma'}$ distribution of graph 2 will be similar to the $\boldsymbol{p}_\Sigma$ distribution of graph 1. For graph 2 we have

$$\boldsymbol{p}_\Sigma = \boldsymbol{p}_{\Sigma'} - \boldsymbol{p}_{\pi_\Sigma}\,,$$

and the $\boldsymbol{p}_\Sigma$ distribution should be more isotropic than the $\boldsymbol{p}_{\Sigma'}$ distribution. Hence, graph 2 is unlikely to give the desired anisotropy, and we therefore discard it. (This does not exclude the possibility that the corresponding graph for Λ^0 production may be more important, due to the existence of a $\Lambda\pi$ resonance (7).)

Graph 3 differs from graph 2 in two

(2) *Charged Hyperon Production by* 16 GeV/c *π^--mesons*, CERN-Pisa-Trieste collaboration, *Phys. Rev. Lett.*, **6**, 303 (1961).

(3) S. Z. BELENLIJ *et al.*: *Fortsch. d. Phys.*, **6**, 524 (1958).

(4) G. F. CHEW and F. E. LOW: *Phys. Rev.*, **113**, 1640 (1959).

(5) F. BONSIGNORI and F. SELLERI: *Nuovo Cimento*, **15**, 465 (1960).

(6) F. SALZMAN and G. SALZMAN: *Phys. Rev. Lett.*, **5**, 377 (1960). See even *Proc. 1960 Ann. Int. Rochester Conf.*

(7) M. ALSTON *et al.*: *Phys. Rev. Lett.*, **5**, 520 (1960).

points: Firstly, it involves the $\pi\pi$ interaction at the mp vertex. Secondly, the transition probability contains the factor $1/(\Delta^2+m_\pi^2)^2$ (instead of $1/(\Delta^2+m_K^2)^2$ for graphs 1 and 2, Δ^2 = invariant momentum transfer), which gives rise to a strong backward peak for $\boldsymbol{p}_{\Sigma'}$, now defined as

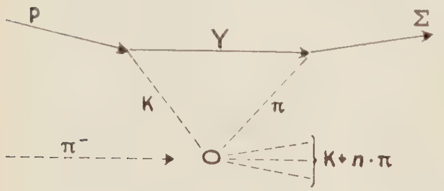
$$\boldsymbol{p}_{\Sigma'} = \boldsymbol{p}_\Sigma + \boldsymbol{p}_K \,.$$

Hence, one may expect that even the Σ backward peak will be stronger for graph 3 that for graph 2.

The Σ^+/Σ^- ratio of graph 3 may be estimated to be 1.8 [8]. From charge conservation, graph 3 gives $r_-=0$. This is in sharp contrast to the r_- values for all the other graphs, including graphs 4 and 5, as we shall see. It may be mentioned that graph 3 contains one more selection rule, namely the transitions to odd n are forbidden by G parity.

3. – Graphs with two virtual bosons.

We now turn to graph 4, which should be important if graph 1 is important. The trouble is that graph 4



Graph 4.

is difficult to calculate. Therefore we propose a model, by saying that graph 4 represents a « final state interaction » of graph 1. The complete model is as follows:

1) A « primary » hyperon Y is produced via graph 1, with « primary cross section » G.

2) After that, Y may absorb one of the pions produced in the mp vertex, and become a real Σ. The « absorption probability » is called H.

3) The final cross section from graph 4 is then $G\cdot H$, and the cross section from graph 1 is $G\cdot(1-H)$.

It is true that the concept of final state interaction cannot be quite correct, since the process $Y+\pi\to\Sigma$ requires that Y and π cannot be both on their mass shells. For an orthodox final state interaction we would require that Y and π should be so close to their mass shells, that scattering will be the dominant process. On the other hand, if the Y hyperon is a Λ^0, only a small amount of « virtuality » is needed. If Y is a Σ, then scattering may be more important. This will bring the r values calculated below closer to 1, so that our main conclusion is not altered. For the charge branching ratios, we assume that the charge of the absorbed pion may be obtained from statistical theory for the mp vertex.

In order to get the r values, one then has to calculate first the probabilities for complete charge distributions (including the pion charges) for graph 1. For instance, the probability for $Y^0K^+\pi^+\pi^-\pi^-\pi^0$ is found to be 0.358 G. From this state, one pion will be absorbed with probability H. The probability for π^+ absorption (to give a final state $\Sigma^+K^+\pi^-\pi^-\pi^0$) will then be $(1/4)\,H$, for π^- absorption $(2/4)\,H$, and for π^0 absorbtion $(1/4)\,H$. A complication is brought in by the fact the Y^0 particle may be a Λ^0 or a Σ^0. Hence, we have to distinguish between two « primary cross sections », G_Λ and G_Σ, and three different absorption probabilities, $H_{\Lambda\Sigma}$, $H_{\Sigma\Lambda}$ and $H_{\Sigma\Sigma}$. Isospin conservation implies

$$G_{\Sigma^+} = 2G_{\Sigma^0}\,, \qquad H_{\Sigma^0\Sigma^0} = 0\,.$$

[8] H. Rollnik: *CERN*, private communication.

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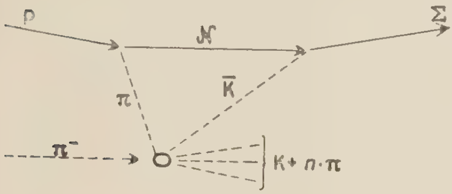
The other H's are charge independent. (It should be noted that such a treatment neglects interference terms, arising from total isospin conservation. The correct isospin formalism for graphs like 4 and 5 will be given in a separate paper. For our present purpose, the above treatment should suffice.)



Graph 5.

As the result of our calculations, we obtain

$$r_- = 0.84 \pm 0.13 \,, \qquad 1.5 < r_+ < 2 \,.$$

The value for r_- is obtained without any assumptions on the G and H parameters, whereas the limits for r_- correspond to a wide range of G and H. For the K^0/K^+ ratio in Λ^0 and Σ^0 production, we find $r_0 \approx 1$.

Finally, we consider graph 5, which gives a branching ratio $\Sigma^+/\Sigma^- < 0.5$. Here probably one should allow one pion to be created together with the Σ, in which case the Σ^+/Σ^- ratio will approach 1. In both cases, we obtain r_- not much smaller than 1, so an r_- measurement probably cannot distinguish between these graphs and graph 4.

We add a remark on « central » collisions: they are expected to yield an isotropic Σ angular distribution, and $\Sigma^+/\Sigma^- \sim 0.9$. That means, the *percentage* of the isotropic « background » for Σ^- creation will be twice as large as for Σ^+ creation.

4. – $K\pi$ resonance.

The existence of a $K\pi$ resonance at total energy 878 MeV has been suggested (9). Treating this resonance as a particle K', one may insert K' instead of K as the virtual boson in graph 1. If K' is taken to be a scalar, isospin 1/2 particle, it gives rise to a stronger backward peak than the pseudoscalar K, despite its larger mass.

On the other hand, K' may be present among the particles from the mp vertex (except for graph 3, of course). The K' having isospin 1/2, the kaon from the « decay » $K' \to K + \pi$ will have the same charge as the K' only in 33% of the cases. This will have an unpleasant influence on the r values, as it will diminish the original kaon charge difference. Thus, a small admixture of K' will drive down r_+ for graph 4.

* * *

The author thanks Professor O. KLEIN for the hospitality extended to him at the Institute for Theoretical Physics, University of Stockholm. Thanks are also due to the colleagues at the Institute for stimulating discussions, and to Dr. S. NILSSON, CERN, for information on the details of the 16 GeV/c experiment.

(9) M. H. ALSTON, L. W. ALVAREZ, PH. EBERHARD, M. L. GOOD, W. GRAZIANO, H. K. TICHO and S. G. WOJCICKI: *Phys. Rev. Lett.*, **6**, 300 (1961).

Equilibrium Families in the Liquid Drop Model.

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(ricevuto il 23 Maggio 1961)

Recently [1] Cohen and Swiatecki have drawn attention on the existence of distinct series of equilibrium shapes of a nucleus treated according to the classical liquid drop model. They have also given a semiquantitative classification of these shapes and indicated their importance in connection with the theory of fission.

We have thought it useful to investigate whether this problem could be treated quantitatively using formulae already derived in previous works [2] and obtain, at least in a first approximation, the interrelation between the different equilibrium series.

To that purpose, limiting ourselves to the case of P_2 deformations superimposed on the basic ellipsoid, we have put the potential energy of the nucleus in the form

$$u(y, \alpha_2) = \left[\bar{A}(y) - 1 + 2x(A(y) - 1)\right] + \alpha_2\left[\bar{B}_2(y) + 2xB_2(y)\right] + \alpha_2^2\left[\bar{C}_{22}(y) + 2xC_{22}(y)\right], \tag{1}$$

where the notations are those already adopted in [2]. From the equilibrium conditions

$$\left\{\begin{aligned} &\frac{\partial u}{\partial y} = 0, \\ &\frac{\partial u}{\partial \alpha_2} = 0, \end{aligned}\right. \tag{2}$$

we obtain, by elimination of α_2, an equation:

$$\Phi(y, x) = 0, \tag{3}$$

(1) C. Cohen and W. J. Swiatecki: *The deformation energy of a charged drop IV. Evidence for a discontinuity in the conventional family of saddle point shapes*, January 1961. Report of the Physics Institute of the University of Aarhus (Authors private communication).

(2) U. L. Businaro and S. Gallone: *Nuovo Cimento*, **1**, 629, 1277 (1955); **5**, 315 (1957).

of the third degree in x. There will thus exist, in general, for each excentricity y of the basic ellipsoid, three equilibrium shapes corresponding to x parameters given by the roots of (3).

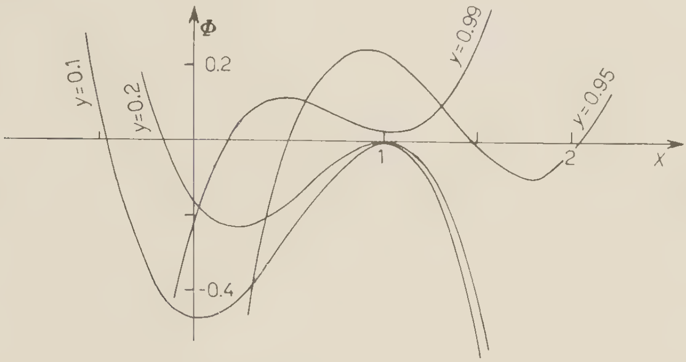

Fig. 1. – Behaviour of the function $\Phi(y, x)$.

The behaviour of the function $\Phi(y, x)$ is indicated in Fig. 1. For vanishing excentricities, we have:

$$\Phi(0, x) = (x - 1)^2 (x + \tfrac{1}{2}) , \tag{4}$$

which indicates the merging of two equilibrium families in the limit $y \to 0$.

Fig. 2 indicates, as a function of y, the x values corresponding to equilibrium configurations. This figure should be compared with Fig. 39 of the work of SWIATECKI and COHEN in which the parameter $R_{\max}/R_0$ plays the same rôle as our y.

It is interesting to note the appearence of a series of shapes very close to ellipsoids (branch y_1 of Fig. 2).

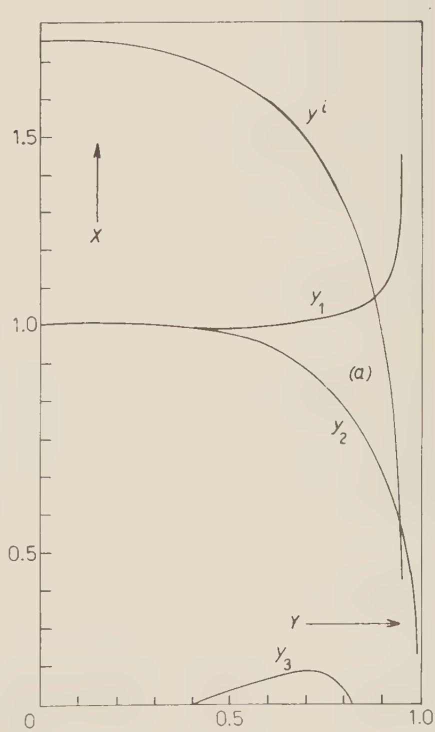

Fig. 2. – y_2 represents the family of Bohr-Wheeler and Frankel-Metropolis saddle shapes. y_1 represents the family studied by Weizsäcker and Wick. The projection of the asymmetric series in the (y, x) plane is indicated by the portion (a) of the inversion curve y^i.

These shapes had already been studied by WEIZSÄCKER (3) and WICK (4) in connection with the hypothesis of WEFELMEYER (5) that under the action of Coulomb forces the very heavy nuclei could differ sensibly from the spherical shape.

As for the existence of shapes of the non symmetric type (2) it could be seen that the « inversion » curve (2), which indicates the appearence of an instability in the P_3 mode, intersects both series y_1 and y_2.

The asymmetric family is in fact contained within the limits $x \sim 0.47$ and $x \sim 1.67$.

* * *

The author wishes to acknowledge the assistance of Miss. M. ANGELINI and Mr. P. MARIANI who performed the numerical computations. Further calculations are in hand to take into account deformations of the P_4 type.

Note added in proof.

The interpretation of the y_3 series is puzzling. The values of α_2 associated with this solution are $\alpha_2 \sim - y/3$, showing that the corresponding shapes are an approximation to the spherical configuration of equilibrium. y_3 is thus not related to the Frankel-Metropolis two tangent spheres type family. I am indebted to W. J. SWIATECKI for this remark.

(3) C. F. VON WEIZSÄCKER: *Naturwiss.*, **27**, 133 (1939).
(4) G. C. WICK: *Nuovo Cimento*, **16**, 229 (1939).
(5) W. WEFELMEYER: *Naturwiss.*, **27**, 110 (1939).

Pion Pion Interaction and Proton Antiproton Annihilation at Rest.

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(ricevuto il 17 Giugno 1961)

Three main points have been recently discussed concerning the pion annihilation of nucleon-antinucleon pairs at rest, namely:

a) the validity of the Day, Snow and Sucher (DSS) argument in favour of absorption from S initial states;

b) the indication on the spin state (singlet or triplet) of the annihilating pair, derived from the observed pion states;

c) the influence of a $T=J=1$ resonant pion-pion state on the angular and energy distributions of the produced pions.

These three points have been in so far discussed separately. It is the purpose of this note to point out that more information could be obtained by discussing them together, as they are not independent and as the solution of one of them can give a clue to the understanding of the other two problems.

Let us briefly summarize what has been done up to now:

Point *a*): B. P. Desai (1), by using a specific model (the Ball-Chew model (2)), has been able to confirm the argument of DSS (3) when applied to the nucleon-antinucleon annihilation. It seems therefore probable that most of the annihilations at rest take place from S states. A possible experimental check, proposed recently by d'Espagnat (4), requires however the analysis of $K^0\overline{K}^0$ annihilations with no extra pions — a process that is known to represent less than 0.1% of the annihilation processes (5).

Point *b*): Two-pion annihilation at rest seems to be present with a frequency

(1) B. P. Desai: *Phys. Rev.*, **119**, 1385 (1960).

(2) J. S. Ball and G. C. Chew: *Phys. Rev.*, **109**, 1385 (1958); J. S. Ball and J. R. Fulco: *Phys. Rev.*, **113**, 647 (1959).

(3) T. B. Day, G. A. Snow and J. Sucher: *Phys. Rev. Lett.*, **3**, 61 (1959).

(4) B. d'Espagnat: *A proposed experimental test of the DSS argument based on antiproton annihilation into kaons*, CERN preprint, 1239 (1961).

(5) As quoted, for instance, by E. O. Okonov: Dubna preprint D-635 (1961).

considerably smaller than that indicated by statistical theories ([6]). This would be the case if the annihilation took place preferentially from singlet states ([7]), and it does not seem necessary to introduce a new selection rule, as that proposed by Shirokov and Okonov ([8]).

Point *c*): This is the most tricky point, as there are in this case no simple invariance or conservation principles and any result will heavily depend on the assumed model for the annihilation. A resonance in the pion-pion scattering amplitude has been introduced by many authors by different techniques into the statistical theory ([9]). The most recent and interesting discussion on the argument has been given by Pinski, Sudarshan and Mahantappa ([10]) and by Pinski ([11]); they propose the use of the reaction $\bar{p}n \to \pi^-\pi^-\pi^+$, where a comparison can be made between the « mass » distribution of the « like pion » pairs (only $T=2$) and that of the « unlike pion » pairs ($T=0, 1, 2$). The latter distribution ought to be sensitive to the presence of a $T=J=1$ pion-pion resonance. An experimental confirmation of this prediction has not been given as yet; previous analyses on the angular correlations between « like » and « unlike » pions have been shown to contain no unambigous indications on the existence of the resonance, as the experimental curves can also be understood in terms of the modifications introduced by the correct application of the Bose statistics to the statistical model ([12]).

We shall now show that a more suitable choice of the annihilation process to be analysed can both improve the sensitivity of the result to the existence of a pion-pion resonance in the $T=J=1$ state (resonance that we shall call henceforth, for simplicity, the « bi-pion », remembering however that it consists really of a Breit-Wigner form for the pion-pion scattering amplitude) and give some useful indications on the most plausible solution of points *a*) and *b*). The argument goes as follows.

Let us discuss the p-$\bar{p}$ annihilation at rest from S or P states); then Table V of ref. ([7]) lists the following final states for the process: $p+\bar{p} \to \pi^+ + \pi^- + \pi^0$ (we remember that the first letter refers to the relative π^+-π^- angular momentum, while the second letter to that of the π^0 with respect to the π^+-π^- c.m.):

TABLE I.

Initial state	Final states
$^1S_0$	$(Ss)_0$, $(Dd)_0$, ...
$^3S_1$	$(Pp)_1$, $(Ff)_1$, ...
$^1P_1$	$(Ps)_1$, $(Pd)_1$, ...
$^3P_0$	
$^3P_1$	$(Sp)_1$, $(Dp)_1$, ...
$^3P_2$	$(Dp)_2$, $(Df)_2$, ...

It is now easy to see from Table I that some of the final states contain the π^+-π^- pair in its pure « bi-pion » configuration; as a matter of fact, states like (Pp) or (Ps) have the π^+-π^- pair in an antisymmetric orbital state, which implies $T=1$. Those are indeed the final states to be studied, if the influence of the « bi-pion » has to be analysed at its best. We believe that it will be reasonable to assume that, should the final

([6]) See, for instance, the paper quoted in ref. ([5]).

([7]) D. AMATI and B. VITALE: *Nuovo Cimento*, **2**, 719 (1955).

([8]) M. SHIROKOV and E. O. OKONOV: *Journ. Exp. Theor. Phys.*, **39**, 1059 (1960).

([9]) E. EBERLE: *Nuovo Cimento*, **8**, 610 (1958); T. GOTÔ: *Nuovo Cimento*, **8**, 625 (1958); F. CERULUS: *Nuovo Cimento*, **14**, 827 (1959).

([10]) G. PINSKI, E. C. G. SUDARSHAN and K. Y. MAHANTAPPA: *Proc. of the 10-th Rochester Conference* (New York, 1960), p. 173.

([11]) G. PINSKI: *Phys. Rev. Lett.*, **6**, 136 (1961).

([12]) G. GOLDHABER, S. GOLDHABER, T. D. LEE and A. PAIS: *Phys. Rev.*, **120**, 300 (1960).

state contain (Pp) or (Ps) configurations, there a « mass » distribution of the π^+-π^- pair (in its c.m.) will be strongly affected by the « bi-pion ». The mass distribution will be affected in a similar but by far stronger way than in the distributions studied in ref. ([10,11]). Conversely, should this « mass » distribution show nothing of the expected behaviour, it will be reasonable to assume that the above-mentioned states were not present among the final states.

The implications of this result on points a) and b) are now evident. If the « mass » distribution shows the « bi-pion » effect, then the initial states are either $^3S_1$ or $^1P_1$ states, the former leading to a $(Pp)_1$ final state (of course, there will be a small contribution of higher angular momentum states like $(Ff)_1$, but this will only add to the statistical background), the latter to a (Ps) state. But an analysis in terms of the angle α between the π^0 momentum with respect to the π^+-π^- c.m. and the direction of relative π^+-π^- motion, could easily dispose of this ambiguity as a (Pp) configuration leads to a $\cos^2\alpha$ distribution and a (Ps) configuration leads to an isotropic distribution. An absorption from $^3S_1$ initial states will then confirm the DSS argument but rule out the possibility of explaining the two-pion annihilation damping in terms of singlet annihilation. An absorption from $^1P_1$ states will rule out the DSS argument but the « singlet » argument of point b) will still hold.

If on the contrary the proposed « mass » distribution does not show the expected behaviour, according to our assumption we have to conclude that the annihilation takes place mainly from $^1S_0$ or (and) $^3P_{1,2}$ states. As the first possibility agrees both with the DSS argument and with the « singlet » interpretation of point b), we believe that this should be assumed as the most probable result of the analysis.

Let us now summarize our proposal: Take $\mathrm{p}+\bar{\mathrm{p}}\to\pi^++\pi^-+(?^0)$ at rest; measure the momenta p_+, p_- and the angle ϑ between $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$. Extract from the statistics only those events where the $(?^0)$ system consists only of one π^0, which can be done by requiring consistency of the measured quantities with the relation:

$$(1)\qquad \mu^2 = 4m^2 + t - 4m(E_+ + E_-)\,,$$

where

$$E_\pm = (p_\pm^2 + \mu^2)^{\frac{1}{2}}\,,$$

and t is given by

$$(2)\qquad t = 2(\mu^2 + E_+E_- - p_+p_-\cos\theta)\,,$$

and represents the « mass » squared of the π^+-π^- in their c.m. Plot now this distribution in t. The « bi-pion » effect should present itself as a well marked peak in correspondence of the « bi-pion » squared mass, namely around $(20\div22)\mu^2$. If the effect is absent, then conclude that the annihilation took place from $^1S_0$ (more probable) or $^3P_{1,2}$ (less probable) initial states. If the effect is present then calculate the quantity $P(\alpha)\,\mathrm{d}\alpha$, with

$$(3)\qquad \cos\alpha = (2m/q)(t-4\mu^2)^{-\frac{1}{2}}(E_- - E_+)\,,$$

where

$$q^2 = \frac{1}{4t}\left[(4m^2 - t - \mu^2)^2 - 4\mu^2 t\right],$$

and plot it. A $\cos^2\alpha$ distribution will lead to the conclusion that the annihilation took place from a $^3S_1$ initial state; an isotropic distribution will make highly probable that the initial state was a $^1P_1$ state.

* * *

One of us (V.D.A.) wants to thank warmly the Istituto di Fisica dell'Università di Roma for the offered hospitality.

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A. D. GALANIN – *Thermal Reactor Theory*. Pergamon Press, London, 1960; pp. XIV-412; £ 5.

Scritto da uno dei più autorevoli fisici russi dei reattori, il libro si rivolge agli studiosi già introdotti nello studio di questa materia, della quale fornisce un quadro organico e logicamente collegato. In esso viene trattato quanto specificamente concerne il calcolo neutronico del reattore, mentre non vengono affrontati argomenti particolari, come ad es. i problemi di schermaggio.

Purtroppo il volume, scritto nel 1952 ed aggiornato con una II edizione nel 1958, prima della II Conferenza di Ginevra, giunge a noi solo ora, nella traduzione inglese. Non si tratta quindi di un libro modernissimo e le date citate definiscono sufficientemente i limiti di validità di questa affermazione.

Pur non mancando oggi libri di testo pregevoli sulla fisica dei reattori termici, tuttavia la lettura del volume del Galanin è particolarmente interessante soprattutto quando viene esposto il cosiddetto « punto di vista russo ». Ad esempio, nella teoria del reattore eterogeneo, mentre il fattore di fissione veloce ε viene trattato in modo simile a quello noto, ad es., dal libro di Glasstone ed Edlund, nel calcolo del fattore p — probabilità di fuga alla risonanza viene esposta la teoria di Gurevich e Pomerančuk nella quale l'analisi del percorso del neutrone viene svolta tenendo conto degli effetti geometrici e dello scattering ed assorbimento nel moderatore, ma ignorando lo scattering ed il rallentamento entro gli elementi di materiale fertile, se sottili.

Accanto alla trattazione del reattore eterogeneo, merita menzione quella dello spettro neutronico termico, in cui viene affrontato il problema della giunzione tra lo spettro maxwelliano e lo spettro $1/E$ nel caso di un moderatore gassoso monoatomico e nel caso di un moderatore pesante.

Pregio del libro è il costante intendimento dell'autore di fornire i mezzi pratici per il calcolo. Infatti spesso viene data la valutazione numerica delle formule ottenute e vengono indicati gli errori rispetto a trattazioni analoghe.

M. MARSEGUERRA

C. H. WILTS – *Principles of Feedback Control*. Addison - Wesley, Reading, Mass., 1960; pp. X-265; $ 9.75.

Come lo stesso autore dichiara nella prefazione, questo libro è destinato agli studenti dei corsi del California Institute of Technology, ma esso può considerarsi molto utile per coloro che desiderano studiare i principi generali dei controlli.

Il tono del libro è elementare e non presuppone speciali conoscenze: è sufficiente una generica preparazione matematica, soprattutto nel campo delle funzioni di variabili complesse e delle equazioni differenziali ordinarie.

Il volume è articolato in 12 capitoli. I primi sei capitoli sono dedicati ai principi generali dei controlli e della reazione; notevole peso è dato al problema della stabilità del sistema: sono esaminati con dettaglio i criteri di stabilità di Routh-Hurwitz e di Nyquist. Dal capitolo 7 al capitolo 10 si tratta delle tecniche elementari di compensazione, di sistemi di controllo con molti anelli di reazione, e delle rispettive risposte e stabilità.

L'undecimo capitolo è dedicato al controllo dei sistemi campionati. È introdotta la trasformata di Laplace di un segnale campionato. L'argomento, di notevole interesse, avrebbe forse meritato una maggiore estensione.

Il dodicesimo ed ultimo capitolo tratta dell'analisi non lineare.

Il volume offre un quadro abbastanza completo dei principi del controllo da un punto di vista teorico, ma una maggiore completezza si sarebbe avuta se fossero stati introdotte applicazioni ed esempi pratici.

B. Rispoli

J. Orear – *Fundamental Physics*. John Wiley & Sons, New York e London, 1961; 381 pagine, 303 illustrazioni.

Estremamente adatto alla didattica. È scritto da un Fisico della nuova generazione, con stile disinvolto, semplice ed elegante, proprio di un intelligente allievo di Fermi. È un libro adatto a persone di media cultura e potrebbe essere il migliore fra quelli esistenti oggi per la Fisica dei Licei.

La presentazione tipografica è eccellente: la densità dei caratteri di stampa su ogni pagina è metà di quella normale; le illustrazioni abbondano; si usano i colori nero e rosso per le figure geometriche e per i grafici.

Il lettore viene messo subito di fronte al mondo della Fisica: nel capitolo introduttivo si presenta un esempio sulla relatività speciale e uno sulla natura ondulatoria dei corpuscoli.

Le leggi fondamentali sono illustrate non solo da esempî che ormai sono entrati in uso comune in ogni testo di Fisica, ma anche da innumerevoli altri tratti dalla Fisica moderna. Ad esempio nel paragrafo sulla conservazione della quantità di moto si mostra nella stessa pagina l'immagine stroboscopica dell'urto di due sfere accanto ad una fotografia di successivi urti protone-protone ottenuta mediante camera a bolle.

Contiene oltre ai capitoli classici che si trovano su ogni testo del genere, altri dedicati alla Relatività speciale e generale, alla Teoria dei quanti, alla Teoria atomica, alla Fisica Nucleare, alla Fisica delle particelle elementari (decadimento β ed interazioni deboli, antimateria, non conservazione della parità).

L'Autore riesce ad inquadrare infine l'aspetto umanistico, sociale e politico di questo ramo dello scibile.

Per concludere, un libro affascinante per il non fisico e piacevole per il fisico. Ottima la bibliografia.

A. Turrin

D. S. Billington and J. H. Crawford jr. – *Radiation Damage in Solids*. Princeton University Press, Princeton, New Jersey, 1961; pp. XI+450, $ 12.50.

La Sezione di Fisica dei Solidi del Laboratorio di Oak Ridge è uno dei centri più attivi del mondo nel campo delle ricerche sui danni da radiazione prodotti nel reattore nucleare. Questo laboratorio possiede tra l'altro un impianto di refrigerazione che permette di lavorare entro il reattore a temperature prossime a quella dell'elio liquido, condizione questa straordinariamente favorevole per le ricerche fondamentali nella fisica dei solidi irradiati. L'attività del

Laboratorio non si limita però a questo tipo di studi, ma si estende a molti campi collaterali ed in particolare comprende ricerche di tipo applicativo sugli acciai ed altri materiali da costruzione.

Non ci sorprende perciò che, proprio dal Centro di Fisica dei Solidi di Oak Ridge, venga pubblicato un volume che raccoglie e compendia gli studi sui danni da radiazione, ad opera del Direttore D. S. BILLINGTON e del suo stretto collaboratore J. H. CRAWFORD jr.

Senza addentrarsi in trattazioni teoriche di carattere superiore il libro affronta tutti i problemi fondamentali con una esposizione semplice ma « impegnata », all'uso americano.

La bibliografia è notevolmente estesa, ma soffre di qualche lacuna e si può considerare aggiornata a tutto il 1958 e parte del 1959: ciò dovuto indubbiamente alle difficoltà editoriali, che si fanno sentire anche in America.

La materia trattata, suddivisa in undici capitoli, può essere, grosso modo, distinta in due parti: i primi quattro capitoli trattano in generale delle interazioni tra radiazioni e cristalli, dell'influenza dei difetti reticolari sulle proprietà dei solidi e delle sorgenti di radiazioni. Dal quinto capitoli in poi, si espongono i risultati delle esperienze e le loro interpretazioni, ordinati a seconda della natura del materiale irradiato, e cioè nel modo più pratico e razionale. Sono studiati successivamente: metalli, leghe, cristalli covalenti, cristalli ionici, semiconduttori, uranio e grafite.

Nel suo complesso il libro può certo formare una buona base per chi voglia occuparsi del « Radiation Damage », purchè venga completato dalla letteratura più recente ed integrato, ove necessario, con una più profonda trattazione teorica. Ma, qualunque aggiunta possa sembrare desiderabile, rimane a questo testo il pregio insostituibile di illustrare i molti aspetti sperimentali dell'argomento nel modo diretto e concreto che può derivare solo dalla personale esperienza degli Autori e dei loro collaboratori immediati, in un laboratorio come quello di Oak Ridge.

F. A. LEVI

U. INGARD and W. L. KRAUSHAAR – *Introduction to Mechanics, Matter and Waves.* Addison-Wesley, Reading, Mass., London, England, 1960; pp. XV-672, $ 8.75.

Si tratta di uno dei testi che sono stati redatti per i corsi di Fisica Generale al Massachusetts Institute of Technology.

Principio informatore della trattazione è il continuo, ampio, riferimento all'esperienza; è quello cioè di effettivamente presentare la Fisica come la scienza che dai fatti sperimentali risale alle cause che li provocano e li condizionano. Man mano che la trattazione progredisce, ogni nuova grandezza fisica viene introdotta dalla necessità di interpretare in termini fisici il risultato di una esperienza: così, per es., il concetto di forza deriva spontaneamente dalla necessità di misurare la rapidità di variazione temporale della quantità di moto.

L'apparato matematico appare notevolmente alleggerito, rispetto a molta parte dei testi del genere, e, comunque, non risulta mai soverchio; ciò che risulta di notevole importanza pratica nel primo anno dei corsi universitari, allorchè lo studente non è ancora padrone del mezzo matematico. L'esposizione della materia è molto scorrevole e punta direttamente alla sostanza senza divagazioni e pleonasmi. Una abbondante e chiara serie di disegni, e qualche volta di fotografie, illustra e commenta efficacemente il testo. Molti gli esempi di applicazione di metodi generali a problemi e questioni particolari. Al termine di ogni capitolo viene proposta una serie di problemi, per una

metà dei quali la soluzione numerica è fornita nelle ultime pagine del libro.

Dei 23 capitoli in cui il libro è suddiviso, 13 sono dedicati alla meccanica e 7 alla termologia, mentre negli ultimi 3 viene dato un cenno abbastanza esteso della meccanica delle onde elastiche.

F. MARIANI

J. H. SANDERS – *The Fundamental Atomic Constants*. Oxford University Press, 1961.

Ancora un nuovo libro sulle costanti atomiche. Un'opera su questo argomento dopo le ampie comunicazioni al Congresso Internazionale sulle Costanti fondamentali della Fisica (Torino, 1956), tra le quali ricordiamo quelle di R. T. BIRGE, di J. W. M. DUMOND, di J. A. BEARDEN e J. S. THOMSEN, e dopo il libro di E. R. COHEN, K. M. CROVE e J. W. M. DUMOND, *The Fundamental Constants of Physics* (1957) e l'articolo ancora di E. R. COHEN e J. W. M. DUMOND comparso sul volume **35** della *Encyclopedia of Physics* di S. FLÜGGE (1957), dimostra l'interesse sempre più vivo sulla metrologia di base, in particolare sull'influenza che essa subisce dalla Fisica Atomica.

L'Autore considera questa sua opera come una breve monografia che ha soltanto l'intenzione di mettere il punto sulla situazione attuale e si direbbe che, sia per concisione che per chiarezza, raggiunga quanto si propone.

Dopo un cenno sulla scelta delle costanti fondamentali della Fisica: carica dell'elettrone, massa dell'elettrone, massa del protone, velocità delle radiazioni elettromagnetiche nel vuoto, costante di Planck, numero di Avogadro, costante di Boltzmann, costante di gravitazione universale, dedica un capitolo più esteso alla rassegna delle prime misurazioni delle costanti atomiche terminandolo con una utile derivazione dei valori più corretti che da queste misurazioni possono trarsi.

Un ulteriore capitolo l'Autore lo dedica alla costante per eccellenza, la velocità c delle radiazioni elettromagnetiche nel vuoto, la costante che appare nella relazione tra le corrispondenti unità delle grandezze elettromagnetiche secondo che si impieghi il sistema di misure detto elettrostatico e quello detto elettromagnetico, tra unità di massa ed unità di energia. Questa sola costante, famosissima tra tutte le altre, può, secondo l'Autore, venire misurata con maggiore precisione con metodi diretti che con metodi indiretti (precisione 1 su 10^6).

Invero a noi sembra che per altre misure fondamentali della Fisica stia avvenendo la stessa cosa; perchè pensiamo che analogo sia il caso della determinazione della costante di gravitazione universale in un luogo. Con il metodo di Volet della caduta libera nel vuoto sembra che anche questa determinazione possa raggiungere la precisione di circa 1 su 10^6, mentre è noto che i metodi classici non giungono a tanto.

È da notare che l'Autore, dopo aver esaminato le prime misurazioni di c, i lavori sperimentali compiuti nel periodo dopo il 1941 e i metodi recenti (interferometria nel campo delle microonde secondo Froome, 1954-1958, metodo di Bergstrand, 1949-1957), conclude con una previsione critica su quanto può attendersi da rinnovate misure su questa costante. Non vi è dubbio che ad accrescere la precisione della misura di c si rischia di ottenere un risultato illusorio a causa, secondo l'Autore, della indeterminazione della unità di lunghezza. Ma i lavori che si sono conclusi con l'adozione del metro ottico da parte della XI Conferenza Generale dei Pesi e Misure (Ottobre 1960), ci assicura che ormai l'unità di lunghezza può considerarsi definita entro circa 2 su 10^9. Il timore espresso da Sanders non ci appare quindi giustificato.

Il quarto capitolo è dedicato a recenti misurazioni riguardanti: il rapporto giromagnetico del protone, il momento magnetico del protone, la massa del protone, ecc. Vi si fa una rassegna dei più recenti lavori (vengono citati lavori pubblicati fino al 1959), basati essenzialmente sulla risonanza nucleare, compiuti nei vari laboratori.

Infine un ultimo capitolo riguarda la determinazione dei valori più attendibili delle costanti atomiche che possono trarsi dalle misure fornite dalle esperienze. L'argomento si basa largamente sui metodi indicati da Cohen e DuMond nell'opera citata. Il Sanders prevede che su questi metodi si addiverrà ad una revisione dei valori attuali.

Alcune Appendici al testo riguardanti i simboli comunemente usati, i campioni delle grandezze fondamentali e infine una tabella delle Costanti completano l'opera di Sanders.

Senza esitazioni questa monografia va giudicata aggiornata e utile. Ma è doveroso osservare che l'Autore ignora praticamente il Sistema Internazionale MKSA °K che appare soltanto nella definizione dell'ampere. Nella monografia di Sanders sopravvivono quindi i vari sistemi C.G.S. di unità e.m.u. ed e.s.u. L'Autore misura H, intensità del campo magnetico in e.m.u. (gauss) (vedi p. 68) mentre è ben noto che H si misura in e.m.u. (Oe). Tra i vantaggi del Sistema Internazionale va annoverato quello della maggior chiarezza; sarà meno facile confondere H, intensità del campo magnetico ($m^{-1} \cdot A$) con B, induzione magnetica ($bes \cdot s^{-2} \cdot A^{-1}$), che non nei sistemi C.G.S. dove gauss e Oe si scambiano facilmente le parti.

F. Demichelis

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